Received: *February 7, 2001*, Accepted: *April 6, 2001*
Revised: *March 16, 2001*

Anholonomic soliton-dilaton and black hole solutions in general relativity

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Abstract: A new method of construction of integral varieties of Einstein equations in three dimensional (3D) and 4D gravity is presented whereby, under corresponding redefinition of physical values with respect to anholonomic frames of reference with associated nonlinear connections, the structure of gravity field equations is substantially simplified. It is shown that there are 4D solutions of Einstein equations which are constructed as nonlinear superpositions of soliton solutions of 2D (pseudo) euclidean sine-Gordon equations (or of lorentzian black holes in Jackiw-Teitelboim dilaton gravity). The Belinski-Zakharov-Meison solitons for vacuum gravitational field equations are generalized to various cases of two and three coordinate dependencies, local anisotropy and matter sources. The general framework of this study is based on investigation of anholonomic soliton-dilaton black hole structures in general relativity. We prove that there are possible static and dynamical black hole, black torus and disk-cylinder like solutions (of non-vacuum gravitational field equations) with horizons being parametrized by hypersurface equations of rotation ellipsoid, torus, cylinder and another type configurations. Solutions describing locally anisotropic variants of the Schwarzschild-Kerr (black hole), Weyl (cylindrical symmetry) and Neugebauer-Meinel (disk) solutions with anisotropic variable masses, distributions of matter and interaction constants are shown to be contained in Einstein's gravity. It is demonstrated in which manner locally anisotropic multi-soliton-dilaton-black hole type solutions can be generated.

Keywords: Solitons Monopoles and Instantons, Classical Theories of Gravity, Black Holes.

JHEP04(2001)009

Contents

1. Introduction

1.1 Soliton-black hole solutions in modern gravity and string/brane theories

One of the main ingredients of the last developments in string theory and gravity is the investigation of connections between black holes and non-perturbative structures of string theory such as Bogomol'nyi-Pasad-Sommerfeld (BPS) solitons or D-branes [8, 31]. New ways to address and propose solutions of old and new fundamental problems of black hole physics are opened via explanation of black hole thermodynamics in terms of microscopic string and membrane physics.

Similar fundamental problems of black hole physics have been recently analyzed in the recent literature using low-dimensional gravity models (see, for instance, ref. [32]). Dilaton gravity theories in two spacetime (2D) dimensions have been used as theoretical laboratories of studding 4D black hole physics.

One of 2D theories of particular interest is the so-called Jackiw-Teitelboim theory (JT) [16] because of its connection to the Liouville-Polyakov action in string theory. The black hole solutions to JT-gravity are dimensional reductions of the Banados-Teitelboim-Zanelli (BTZ) black hole solutions [1] and exhibit usual thermodynamic properties with black hole entropy [21]. Such JT black hole solutions describe spacetimes of constant curvature.

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The aim of this paper is to develop and apply a new method of construction of 4D solutions of the Einstein equations by considering nonlinear superpositions of 2D and 3D soliton-dilaton-black hole metrics which are treated as some non-perturbative structures in general relativity.

1.2 Locally anisotropic soliton, black hole and disk-cylinder like solutions

Mathematicians (see, for instance, the review [30]) have considered for a long time the relationship between euclidean, $\epsilon = 1$ (pseudo-euclidean, or lorentzian, $\epsilon = -1$), 2D metrics

$$ds^2 = \epsilon \sin^2\left(\frac{\phi}{2}\right) dt^2 + \cos^2\left(\frac{\phi}{2}\right) dr^2 \tag{1.1}$$

with the angle $\phi(t, r)$ solving the lorentzian (euclidean) sine-Gordon equation,

$$-\epsilon \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial r^2} = \widetilde{m}^2 \sin \phi\,, \tag{1.2}$$

which determine some 2D rimannian geometries with constant negative curvature,

$$\widetilde{R} = -2\widetilde{m}^2 = \text{const}\,. \tag{1.3}$$

The angle $\phi\,(t, r)$ from (1.1) describes an embedding of a 2D manifold into a three-dimensional (pseudo) euclidean space.

The topic of construction of soliton solutions in gravity theories has a long history (see refs [4, 12, 24, 27]). For 4D vacuum Einstein gravity the problem was tackled by investigating metrics $g_{\alpha\beta}$ of signature $(-, +, +, +)$ (in this paper permutations of sines will be also considered), with a $2 + 2$ spacetime splitting,

$$-ds^2 = g_{ij}\left(x^i\right) dx^i dx^j + h_{ab}(x^i) dy^a dy^b\,, \tag{1.4}$$

where $g_{ij} = \mathrm{diag}[-f(x^i), f(x^i)]$ and $\det h_{ab} < 1$ and the local coordinates are denoted

$$u^\alpha = \left(\{x^i = (x^1, x^2)\}, \{y^a = (y^3 = z, y^4)\}\right),$$

or, in brief, $u = (x, y)$. We adopt the convention that the x-coordinates are provided with Latin indices of type $i, j, k, \ldots = 1, 2$; the y-coordinates are with indices of type $a, b, c, \ldots = 3, 4$ and the 4D u-coordinates will be provided with Greek indices $\alpha, \beta, \ldots = 1, 2, 3, 4$. The meaning of coordinates (space or time like ones) will depend on the type of construction under consideration. Belinski and Zakharov [4] identified y^4 with the time like coordinate; Maison [24] treated y^a as space variables. It was proved that the vacuum Einstein equations, $R_{\alpha\beta} = 0$, are satisfied if the components $h_{ab}(x^i)$ are solutions of a generalized (euclidean) sine-Gordon equation.

In two recent papers [10] Gegenberg and Kunstatter investigated the relationship between black holes of JT dilation gravity and solutions of the sine-Gordon field theory. Their constructions were generalized by Cadoni [6] to soliton solutions of 2D dilaton gravity models which describes spacetimes being of non constant curvature.

In this work we explore the possibility of anholonomic generalizations of the Belinski-Zakharov-Maison [4, 24] soliton constructions (1.4) in a fashion when the coefficients of the matrix h_{ab} could depend on three variables (x^i, z). The matrix $g_{ij}(x^i)$ will be defined by some Gegenberg-Kunstatter-Cadoni 2D soliton-black hole solutions [10, 6], their conformal transforms, or by the factor $f(x^i)$ from (1.4) which could be related with solitons for h_{ab}. We emphasize that the resulting 4D (pseudo) riemannian metrics, with generic local anisotropy (in brief, anisotropic metrics, or anisotropic metrics) will be found to solve both type of vacuum or with energy-momentum source Einstein equations.

For definiteness, we consider 4D metrics parametrized by ansatzs of type

$$g_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b h_{ab} & N_j^e h_{ae} \\ N_i^e h_{be} & h_{ab} \end{bmatrix} \tag{1.5}$$

which are given with respect to a local coordinate basis $du^\alpha = (dx^i, dy^a)$ being dual to $\partial/u^\alpha = (\partial/x^i, \partial/y^a)$. For simplicity, the 2D components g_{ij} and h_{ab} are considered to be some diagonal matrices (for two dimensions a diagonalization is always possible),

$$g_{ij}(x^k) = \begin{pmatrix} g_1(x^k) & 0 \\ 0 & g_2(x^k) \end{pmatrix} \tag{1.6}$$

and

$$h_{ab}(x^k, z) = \begin{pmatrix} h_3(x^k, z) & 0 \\ 0 & h_4(x^k, z) \end{pmatrix}. \tag{1.7}$$

The components $N_i^a = N_i^a(x^i, z)$ will be selected as to satisfy the 4D Einstein gravitational field equations.

The metric (1.5) can be rewritten in a very simple form

$$g_{\alpha\beta} = \begin{pmatrix} g_{ij}(x^k) & 0 \\ 0 & h_{ab}(x^k, z) \end{pmatrix} \tag{1.8}$$

with respect to some $2+2$ anholonomic bases (tetrads, or vierbiends) defined

$$\delta_\alpha = (\delta_i, \partial_a) = \frac{\delta}{\partial u^\alpha} = \left(\delta_i = \frac{\delta}{\partial x^i} = \frac{\partial}{\partial x^i} - N_i^b \left(x^j, y\right) \frac{\partial}{\partial y^b}, \partial_a = \frac{\partial}{\partial y^a} \right) \tag{1.9}$$

and

$$\delta^\beta = \left(d^i, \delta^a\right) = \delta u^\beta = \left(d^i = dx^i, \delta^a = \delta y^a = dy^a + N_k^a \left(x^j, y^b\right) dx^k\right). \tag{1.10}$$

The coefficients $N_j^a(u^\alpha)$ from (1.9) and (1.10) could be treated as the components of an associated nonlinear connection, N-connection, structure (see [3, 26, 33]; in this work we do not consider in details the N-connection geometry).

A specific point of this paper, comparing with another soliton approaches in gravity theories, is to show how anholonomic constructions can be used for generation of 4D soliton-dilaton-black hole non-perturbative structures in general relativity. This way a correspondence between solutions of so-called locally anisotropic (super) gravity and string theories [33] and metrics given with respect to anholonomic frames in general relativity is derived.

Ansatzs of type (1.6), (1.7) and (1.8) can be used for construction of an another class of solutions with generic local anisotropy of the Einstein equations. If the gravitational field equations are written with respect to an anholonomic basis (1.9) and/or (1.10), the coefficients g_{ij}, h_{ab} and N_j^a satisfy some very simplified systems of partial differential equations. We can construct various classes of black hole and disk-cylinder like solutions which in the locally isotropic limit are conformally equivalent to some well known BTZ, Schwarzschild and/or Kerr, Weyl cylindrical and Neugebauer-Meinel disk solutions. In general relativity there are admitted singular (in a point, on unclosed infinite lines or on closed curves such as ellipses, circles) solutions with horizons being described by hypersurface equations for rotation ellipsoid, torus, ellipses and so on. A physical treatment of such nonlinear configurations is to consider values like anisotropic mass, oscillation of horizons, variable interaction constants and gravitational non-linear self polarizations.

1.3 Outline

The paper is organized as follows: section 2 reviews the geometry of anholonomic frames on (pseudo) riemannian spaces and associated nonlinear connection structures. There are defined the basic geometric objects and written the Einstein equations with respect to anholonomic frames split by nonlinear connections.

In section 3, there are considered the general properties and reductions of basic geometric objects and field equations for 4D metrics constructed as nonlinear superpositions of 2D horizontal (with respect to a nonlinear connection structure) metrics, depending on two horizontal coordinates, and of 2D vertical coordinates depending on three (two horizontal plus one vertical) coordinates. It is given a classification of such 4D metrics depending on signatures of 2D metrics and resulting 4D metrics.

In section 4, we prove that the Einstein equations admit soliton like 2D (both for horizontal and vertical components of 4D metrics) and 3D (for vertical components of 4D metrics) solutions. There are examined some classes of integral varieties for the Einstein equations admitting non-perturbative structures generated, for instance, by 2D and 3D sine-Gordon and Kadomtsev-Petviashvili equations. Some exact solutions for locally anisotropic deformations of the sine-Gordon systems are constructed.

Section 5 describes an effective locally anisotropic soliton-dilaton field theory and contains a topological analysis of such models.

Section 6 elucidates the interconnection of locally anisotropic 2D soliton and black hole solutions. Nonlinear superpositions to 4D are considered.

In section 7, we construct 3D black hole solutions with generic local anisotropic. As some simplest examples there are taken configurations when the horizon is parametrized by an ellipse and the possibility of oscillation in time of such horizons is shown.

Section 8 is devoted to the physics of 4D black hole and disk-cylinder solutions with generic local anisotropy. There are analyzed the general properties of metrics describing such solutions and discussed the question of their physical treatment. The construction of singular solutions with various type of horizon hypersurfaces is performed by considering correspondingly the rotation ellipsoid, epllipsoidal cylinder, torus, bipolar and another systems of coordinates. It is shown that in the locally isotropic limit such solutions could be equivalent to some conformal transforms of some static or rotating configurations like for the Schwarzschild-Kerr, Weyl cylindrical and Neugebauer-Meinel disk solutions.

In section 9, some additional examples of locally anisotropic soliton-dilaton-black hole solutions are given. It is illustrated how in general relativity we can construct two soliton non-perturbative structures, proved that nonlinear connections and non-diagonal energy-momentum tensor components can induce Kadomtsev-Petviashvily soliton like solutions and there are considered new types of two and three coordinate soliton-dilaton vacuum gravitational configurations.

Finally, in section 10, we discuss our results and present conclusions.

2. Anholonomic frames on (pseudo) riemannian spaces

We outline the geometric background on anholonomic frames modelling 2D local anisotropies (la) in 4D curved spaces [33] (see refs. [13] and [26] for details on spacetime differential geometry and N-connection structures in vector bundle spaces). We note that a frame anholonomy induces a corresponding local anisotropy. Spacetimes enabled with anholonomic frame (and associated N-connection) structures are also called locally anisotropic spacetimes, in brief anisotropic spacetimes.

In this paper spacetimes are modelled as smooth (i.e. class C^∞) 4D (pseudo) riemannian manifolds $V^{(3+1)}$ being Hausdorff, paracompact and connected and provided with corresponding geometric structures of symmetric metric $g_{\alpha\beta}$ of signature $(-,+,+,+)$ and of linear, in general nonsymmetric, connection $\Gamma^{\alpha}{}_{\beta\gamma}$ defining the covariant derivation D_α satisfying the metricity conditions $D_\alpha g_{\beta\gamma}=0$. The indices are given with respect to a tetradic (frame) vector field $\delta^\alpha=(\delta^i,\delta^a)$ and its dual $\delta_\alpha=(\delta_i,\delta_a)$.

A frame (local basis) structure δ_α (1.10) on $V^{(3+1)}$ is characterized by its anholonomy coefficients $w^{\alpha}{}_{\beta\gamma}$ defined from relations

$$\delta_\alpha\delta_\beta-\delta_\beta\delta_\alpha=w^{\gamma}{}_{\alpha\beta}\delta_\gamma\,. \tag{2.1}$$

The elongation (by N-coefficients) of partial derivatives in the locally adapted partial derivatives (1.9) reflects the fact that on the (pseudo) riemannian spacetime $V^{(3+1)}$ it is modelled a generic local anisotropy characterized by anholonomy relations (2.1) when the anholonomy coefficients are computed as follows

$$\begin{aligned}
w^k{}_{ij} &= 0, w^k{}_{aj} = 0, w^k{}_{ia} = 0, w^k{}_{ab} = 0, w^c{}_{ab} = 0\,, \\
w^a{}_{ij} &= -\Omega^a_{ij}, w^b{}_{aj} = -\partial_a N^b_i, w^b{}_{ia} = \partial_a N^b_i\,,
\end{aligned}$$

where

$$\Omega^a_{ij} = \partial_i N^a_j - \partial_j N^a_i + N^b_i \partial_b N^a_j - N^b_j \partial_b N^a_i$$

defines the coefficients of N-connection curvature, in brief, N-curvature. On (pseudo) riemannian spaces this is a characteristic of a chosen anholonomic system of reference.

A $2+2$ anholonomic structure distinguishes (d) the geometrical objects into horizontal (h) and vertical (v) components. Such objects are briefly called d-tensors, d-metrics and/or d-connections. Their components are defined with respect to anholonomic basis, anisotropic basis, of type (1.9), its dual (1.10), or their tensor products (d-linear or d-affine transforms of such frames could also be considered). For instance, a covariant and contravariant d-tensor Z, is expressed

$$Z = Z^\alpha{}_\beta \delta_\alpha \otimes \delta^\beta = Z^i{}_j \delta_i \otimes d^j + Z^i{}_a \delta_i \otimes \delta^a + Z^b{}_j \partial_b \otimes d^j + Z^b{}_a \partial_b \otimes \delta^a\,.$$

A linear d-connection D on anisotropic space $V^{(3+1)}$,

$$D_{\delta_\gamma} \delta_\beta = \Gamma^\alpha{}_{\beta\gamma}(x,y)\,\delta_\alpha\,,$$

is parametrized by non-trivial h-v-components,

$$\Gamma^\alpha{}_{\beta\gamma} = \left(L^i{}_{jk}, L^a{}_{bk}, C^i{}_{jc}, C^a{}_{bc}\right). \tag{2.2}$$

A metric on $V^{(3+1)}$ with $2+2$ block coefficients (1.8) is written in distinguished form, as a metric d-tensor (in brief, d-metric), with respect an anisotropic base (1.10)

$$\delta s^2 = g_{\alpha\beta}(u)\,\delta^\alpha \otimes \delta^\beta = g_{ij}(x,y)dx^i dx^j + h_{ab}(x,y)\delta y^a \delta y^b\,. \tag{2.3}$$

Some d-connection and d-metric structures are compatible if there are satisfied the conditions

$$D_\alpha g_{\beta\gamma} = 0\,.$$

For instance, a canonical compatible d-connection

$${}^c\Gamma^\alpha{}_{\beta\gamma} = \left({}^cL^i{}_{jk}, {}^cL^a{}_{bk}, {}^cC^i{}_{jc}, {}^cC^a{}_{bc}\right)$$

is defined by the coefficients of d-metric (2.3), $g_{ij}(x,y)$ and $h_{ab}(x,y)$, and by the N-coefficients,

$$
\begin{aligned}
{}^{c}L^{i}{}_{jk} &= \frac{1}{2}g^{in}\left(\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk}\right),\\
{}^{c}L^{a}{}_{bk} &= \partial_b N_k^a + \frac{1}{2}h^{ac}\left(\delta_k h_{bc} - h_{dc}\partial_b N_i^d - h_{db}\partial_c N_i^d\right),\\
{}^{c}C^{i}{}_{jc} &= \frac{1}{2}g^{ik}\partial_c g_{jk},\\
{}^{c}C^{a}{}_{bc} &= \frac{1}{2}h^{ad}\left(\partial_c h_{db} + \partial_b h_{dc} - \partial_d h_{bc}\right).
\end{aligned} \tag{2.4}
$$

The coefficients of the canonical d-connection generalize for anisotropic spacetimes the well known Cristoffel symbols.

For a d-connection (2.2) the components of torsion,

$$
\begin{aligned}
T\left(\delta_\gamma, \delta_\beta\right) &= T^{\alpha}_{\ \beta\gamma}\delta_\alpha\,,\\
T^{\alpha}_{\ \beta\gamma} &= \Gamma^{\alpha}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\gamma\beta} + w^{\alpha}{}_{\beta\gamma}
\end{aligned}
$$

are expressed via d-torsions

$$
\begin{aligned}
&T^{i}_{.jk} = T^{i}_{jk} = L^{i}_{jk} - L^{i}_{kj}\,, \quad T^{i}_{ja} = C^{i}_{.ja}, T^{i}_{aj} = -C^{i}_{ja}\,,\\
&T^{i}_{.ja} = 0, \quad T^{a}_{.bc} = S^{a}_{.bc} = C^{a}_{bc} - C^{a}_{cb}\,,\\
&T^{a}_{.ij} = -\Omega^{a}_{ij}, \quad T^{a}_{.bi} = \partial_b N^a_i - L^{a}_{.bj}, \quad T^{a}_{.ib} = -T^{a}_{.bi}\,.
\end{aligned} \tag{2.5}
$$

We note that for symmetric linear connections the d-torsion is induced as a pure anholonomic effect.

In a similar manner, putting non-vanishing coefficients (2.2) into the formula for curvature

$$
\begin{aligned}
R\left(\delta_\tau, \delta_\gamma\right)\delta_\beta &= R_{\beta\ \gamma\tau}^{\ \alpha}\delta_\alpha\,,\\
R_{\beta\ \gamma\tau}^{\ \alpha} &= \delta_\tau \Gamma^{\alpha}{}_{\beta\gamma} - \delta_\gamma \Gamma^{\alpha}{}_{\beta\delta} + \Gamma^{\varphi}{}_{\beta\gamma}\Gamma^{\alpha}{}_{\varphi\tau} - \Gamma^{\varphi}{}_{\beta\tau}\Gamma^{\alpha}{}_{\varphi\gamma} + \Gamma^{\alpha}{}_{\beta\varphi}w^{\varphi}{}_{\gamma\tau}\,,
\end{aligned}
$$

we can compute the components of d-curvatures

$$
\begin{aligned}
R^{.i}_{h.jk} &= \delta_k L^{i}_{.hj} - \delta_j L^{i}_{.hk} + L^{m}_{.hj}L^{i}_{mk} - L^{m}_{.hk}L^{i}_{mj} - C^{i}_{.ha}\Omega^{a}_{.jk}\,,\\
R^{.a}_{b.jk} &= \delta_k L^{a}_{.bj} - \delta_j L^{a}_{.bk} + L^{c}_{.bj}L^{a}_{.ck} - L^{c}_{.bk}L^{a}_{.cj} - C^{a}_{.bc}\Omega^{c}_{.jk}\,,\\
P^{.i}_{j.ka} &= \partial_k L^{i}_{.jk} + C^{i}_{.jb}T^{b}_{.ka} - \left(\delta_k C^{i}_{.ja} + L^{i}_{.lk}C^{l}_{.ja} - L^{l}_{.jk}C^{i}_{.la} - L^{c}_{.ak}C^{i}_{.jc}\right),\\
P^{.c}_{b.ka} &= \partial_a L^{c}_{.bk} + C^{c}_{.bd}T^{d}_{.ka} - \left(\delta_k C^{c}_{.ba} + L^{c}_{.dk}C^{d}_{.ba} - L^{d}_{.bk}C^{c}_{.da} - L^{d}_{.ak}C^{c}_{.bd}\right),\\
S^{.i}_{j.bc} &= \partial_c C^{i}_{.jb} - \partial_b C^{i}_{.jc} + C^{h}_{.jb}C^{i}_{.hc} - C^{h}_{.jc}C^{i}_{hb}\,,\\
S^{.a}_{b.cd} &= \partial_d C^{a}_{.bc} - \partial_c C^{a}_{.bd} + C^{e}_{.bc}C^{a}_{.ed} - C^{e}_{.bd}C^{a}_{.ec}\,.
\end{aligned}
$$

The Ricci tensor

$$
R_{\beta\gamma} = R_{\beta\ \gamma\alpha}^{\ \alpha}
$$

has the d-components

$$R_{ij} = R^{.k}_{i.jk}\,, \qquad R_{ia} = -{}^{2}P_{ia} = -P^{.k}_{i.ka}\,,$$
$$R_{ai} = {}^{1}P_{ai} = P^{.b}_{a.ib}\,, \qquad R_{ab} = S^{.c}_{a.bc}\,. \tag{2.6}$$

We point out that because, in general, ${}^{1}P_{ai} \neq {}^{2}P_{ia}$, the Ricci d-tensor is non symmetric.

Having defined a d-metric of type (2.3) in $V^{(3+1)}$ we can compute the scalar curvature

$$\overleftarrow{R} = g^{\beta\gamma} R_{\beta\gamma}\,.$$

of a d-connection D,

$$\overleftarrow{R} = G^{\alpha\beta} R_{\alpha\beta} = \widehat{R} + S\,, \tag{2.7}$$

where $\widehat{R} = g^{ij} R_{ij}$ and $S = h^{ab} S_{ab}$.

Now, by introducing the values (2.6) and (2.7) into the Einstein's equations

$$R_{\beta\gamma} - \frac{1}{2} g_{\beta\gamma} \overleftarrow{R} = k \Upsilon_{\beta\gamma}\,,$$

we can write down the system of field equations for anisotropic gravity with prescribed anholonomic (N-connection) structure [26]:

$$\begin{aligned} R_{ij} - \frac{1}{2}\left(\widehat{R} + S\right) g_{ij} &= k\Upsilon_{ij}\,, \\ S_{ab} - \frac{1}{2}\left(\widehat{R} + S\right) h_{ab} &= k\Upsilon_{ab}\,, \\ {}^{1}P_{ai} &= k\Upsilon_{ai}\,, \\ {}^{2}P_{ia} &= -k\Upsilon_{ia}\,, \end{aligned} \tag{2.8}$$

where $\Upsilon_{ij}, \Upsilon_{ab}, \Upsilon_{ai}$ and Υ_{ia} are the components of the energy-momentum d-tensor field $\Upsilon_{\beta\gamma}$ (which includes possible cosmological constants, contributions of anholonomy d-torsions (2.5) and matter) and k is the coupling constant

3. 4D anholonomic superpositions of 2D D-metrics

Let us consider a 4D spacetime $V^{(3+1)}$ provided with a d-metric (2.3) when $g_i = g_i(x^k)$ and $h_a = h_a(x^k, z)$ for $y^a = (z, y^4)$. The N-connection coefficients are restricted to be some functions on three coordinates (x^i, z),

$$N^3_1 = q_1(x^i, z)\,, \qquad N^3_2 = q_2(x^i, z)\,,$$
$$N^4_1 = n_1(x^i, z)\,, \qquad N^4_2 = n_2(x^i, z)\,. \tag{3.1}$$

For simplicity, we denoted the partial derivatives as $= \partial a/\partial x^1$, $a' = \partial a/\partial x^2$, $a^* = \partial a/\partial z$ $\dot{a}' = \partial^2 a/\partial x^1 \partial x^2$, $a^{**} = \partial^2 a/\partial z \partial z$.

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The non-trivial components of the Ricci d-tensor (2.6), for the mentioned type of d-metrics depending on three variables, are

$$R_1^1 = R_2^2 = \frac{1}{2g_1 g_2}\left[-(g_1^{''} + \ddot{g}_2) + \frac{1}{2g_2}\left(\dot{g}_2^2 + g_1^{'} g_2^{'}\right) + \frac{1}{2g_1}\left(g_1^{'\,2} + \dot{g}_1 \dot{g}_2\right)\right]; \tag{3.2}$$

$$S_3^3 = S_4^4 = \frac{1}{2h_3 h_4}\left[-h_4^{**} + \frac{1}{2h_4}(h_4^*)^2 + \frac{1}{2h_3} h_3^* h_4^*\right]; \tag{3.3}$$

$$P_{31} = \frac{q_1}{2}\left[\frac{h_4^{**}}{h_4} - \frac{(h_4^*)^{\ 2}}{2h_4^{\ 2}} - \frac{h_3^* h_4^*}{2h_3 h_4}\right] + \frac{h_4^*}{4h_4}\left[\frac{\dot{h}_3}{2h_3} h_4^* + \frac{\dot{h}_4}{2h_4} h_4^*\right] - \frac{\dot{h}_4^*}{2h_4}, \tag{3.4}$$

$$P_{32} = \frac{q_2}{2}\left[\frac{h_4^{**}}{h_4} - \frac{(h_4^*)^{\ 2}}{2h_4^{\ 2}} - \frac{h_3^* h_4^*}{2h_3 h_4}\right] + \frac{h_4^*}{4h_4}\left[\frac{h_3^{'}}{2h_3} h_4^* + \frac{h_4^{'}}{2h_4} h_4^*\right] - \frac{h_4^{'*}}{2h_4}, \tag{3.5}$$

$$P_{41} = -\frac{h_4}{2h_3} n_1^{**} + \frac{1}{2h_3}\left(\frac{h_4}{h_3} h_3^* - \frac{3}{2} h_4^*\right) n_1^*, \tag{3.6}$$

$$P_{42} = -\frac{h_4}{2h_3} n_2^{**} + \frac{1}{2h_4}\left(\frac{h_4}{h_3} h_3^* - \frac{3}{2} h_4^*\right) n_2^*.$$

The curvature scalar $\overleftarrow{R}$ (2.7) is defined by two non-trivial components $\widehat{R} = 2R_1^1$ and $S = 2S_3^3$.

The system of Einstein equations (2.8) transforms into

$$R_1^1 = -\kappa \Upsilon_3^3 = -\kappa \Upsilon_4^4, \tag{3.7}$$

$$S_3^3 = -\kappa \Upsilon_1^1 = -\kappa \Upsilon_2^2, \tag{3.8}$$

$$P_{3i} = \kappa \Upsilon_{3i}, \tag{3.9}$$

$$P_{4i} = \kappa \Upsilon_{4i}, \tag{3.10}$$

where the values of R_1^1, S_3^3, P_{ai}, are taken respectively from (3.2), (3.3), (3.4) and (3.6).

By using the equations (3.9) and (3.10) we can define the N-coefficients (3.1), $q_i(x^k, z)$ and $n_i(x^k, z)$, if the functions $h_i(x^k, z)$ are known as solutions of the equations (3.8).

Now, we discuss the question on possible signatures of generated 4D metrics. There are three classes of anisotropic solutions:

1. The horizontal d-metric is fixed to be of lorentzian signature, $sign\,(g_{ij}) = (-,+)$, the vertical one is of euclidean signature, $sign\,(h_{ab}) = (+,+)$, and the resulting 4D metric $g_{\alpha\beta}$ will be considered of signature $(-,+,+,+)$. The local coordinates are chosen $u^\alpha = (x^1 = t, x^2, y^3 = z, y^4)$, where t is the time like coordinate and the d-metrics are parametrized

$$g_{ij}\left(t, x^1\right) = \begin{pmatrix} g_1 = -\exp a_1 & 0 \\ 0 & g_2 = \exp a_2 \end{pmatrix} \tag{3.11}$$

and

$$h_{ab}\left(t, x^1, z\right) = \begin{pmatrix} h_3 = \exp b_3 & 0 \\ 0 & h_4 = \exp b_4 \end{pmatrix}. \tag{3.12}$$

The energy-momentum d-tensor for the Einstein equations (2.8) could be considered in diagonal form

$$\Upsilon_{\beta}^{\alpha} = \mathrm{diag}[-\varepsilon, p_2, p_3, p_4] \tag{3.13}$$

if the N-coefficients $N_i^a(t, x^1, z)$ are chosen to make zero the non-diagonal components of the Ricci d-tensor (see (3.4) and (3.6)). Here we note that on anisotropic spacetimes, with respect to anholonomic frames, there are possible nonzero values of pressure, $p \neq 0$, even $\varepsilon = 0$.

2. The horizontal d-metric is fixed to be of euclidean signature, $sign\,(g_{ij}) = (+,+)$, the vertical one is of lorentzian signature, $sign\,(h_{ab}) = (+,-)$ and the resulting 4D metric $g_{\alpha\beta}$ will be considered to be a static one with signature $(+,+,+,-)$. The local coordinates are chosen $u^{\alpha} = (x^1, x^2, y^3 = z, y^4 = t)$ and the d-metrics are parametrized

$$g_{ij}\left(x^k\right) = \begin{pmatrix} g_1 = \exp a_1 & 0 \\ 0 & g_2 = \exp a_2 \end{pmatrix} \tag{3.14}$$

and

$$h_{ab}\left(x^k, z\right) = \begin{pmatrix} h_3 = \exp b_3 & 0 \\ 0 & h_4 = -\exp b_4 \end{pmatrix} \tag{3.15}$$

The energy-momentum d-tensor for the Einstein equations (2.8) could be considered in diagonal form

$$\Upsilon_{\beta}^{\alpha} = \mathrm{diag}[p_1, p_2, p_3, -\varepsilon] \tag{3.16}$$

if the coefficients $N_i^a(x^i, z)$ are chosen to diagonalize the Ricci d-tensor (when (3.4) and (3.6) are zero).

3. The horizontal d-metric is fixed to be of euclidean signature, $sign\,(g_{ij}) = (+,+)$, the vertical one is of lorentzian signature, $sign\,(h_{ab}) = (-,+)$. The local coordinates are chosen $u^{\alpha} = (x^1, x^2, y^3 = z = t, y^4)$ and the d-metrics are parametrized

$$g_{ij}\left(x^k\right) = \begin{pmatrix} g_1 = \exp a_1 & 0 \\ 0 & g_2 = \exp a_2 \end{pmatrix} \tag{3.17}$$

and

$$h_{ab}\left(x^k, t\right) = \begin{pmatrix} h_3 = -\exp b_3 & 0 \\ 0 & h_4 = \exp b_4 \end{pmatrix}. \tag{3.18}$$

The energy-momentum d-tensor for the Einstein equations (2.8) is considered in diagonal form

$$\Upsilon_{\beta}^{\alpha} = \mathrm{diag}[p_1, p_2, -\varepsilon, p_4] \tag{3.19}$$

if the N-coefficients $N_j^a(x^i, t)$ make zero the non-diagonal components of the Ricci d-tensor (with vanishing (3.4) and (3.6)).

The following sections are devoted to a general study and explicit constructions of 4D solutions of the Einstein equations via type 1-3 nonlinear superpositions of 2D soliton-dilaton-black hole d-metrics.

4. Locally anisotropic soliton like equations

We have found in the last section that the vertical component of energy-momentum d-tensor is the non-vacuum source of the horizontal components of a d-metric (see the equations (3.7)) and, inversely, following (3.8), one could conclude that the horizontal component of energy-momentum d-tensor is the non-vacuum source of the vertical components of a d-metric. The horizontal and vertical components of the distinguished Einstein equations (2.8) are rather different by structures and this is taken into account by choosing the metric ansatz (1.5) which gives rise in a very simplified form of partial differential equations. If the 2D h-metric depends on two variables, $g_{ij} = g_{ij}(x^k)$, with the diagonal components satisfying a second order partial differential equation with respect to 'dot' and 'prime' derivatives, the 2D v-metric could be on three variables, $h_{ab} = h_{ab}(x^k, z)$, with diagonal components satisfying a second order partial derivation with respect to 'star' derivatives. The purpose of this section is to prove that both type of Einstein h- and v-equations admit soliton like solutions.

4.1 Horizontal deformed sine-Gordon equations

Let us parametrize the horizontal part of d-metric (h-metric) as

$$g_1 = \epsilon \sin^2 \left[v\left(x^i\right)/2 \right], \epsilon = \pm 1\,,$$
$$g_2 = \cos^2 \left[v\left(x^i\right)/2 \right].$$

The non-trivial component of the Ricci d-tensor (3.2) is written

$$\epsilon R_1^1 = \frac{1}{\sin v}\left(\ddot{v} - \epsilon v''\right) + \rho\left(x^i\right), \tag{4.1}$$

where

$$\rho\left(x^i\right) = \rho\left(v, \dot{v}, v'\right) = \frac{\cos v - 1}{\sin^2 v}\left(\dot{v}^2 - \epsilon v'^{\,2}\right). \tag{4.2}$$

The horizontal Einstein equations (3.7) are

$$\left(\ddot{v} - \epsilon v''\right) + \frac{\cos v - 1}{\sin v}\left(\dot{v}^2 - \epsilon v'^{\,2}\right) = \epsilon\kappa\Upsilon_3^3\ \sin v\,, \tag{4.3}$$

being compatible for matter states when $\Upsilon_3^3 = \Upsilon_4^4$. For simplicity, we consider the case of constant energy density or pressure (depending on the type of fixed signature), $\Upsilon_3^3 = \Upsilon_3 = \text{const}$. The equation (4.3) defines some components of a 4D metric (1.8) and is defined by a locally anisotropic deformation of the euclidean (for $\epsilon = -1$), or lorentzian (for $\epsilon = -1$), sine-Gordon equation (1.2), which are related with 2D (pseudo) riemannian metrics (1.1) and constant curvatures (1.3).

The first term from (4.1) transforms into a negative constant, $-\widetilde{m}^2$, if the function $v(x^i)$ is chosen to be a soliton type one which solves the sine-Gordon equation (1.2). The physical interpretation of terms depends of the type of the solutions we try to construct (see below). We note that if $\epsilon = -1$, the second term, $\rho(x^i)$, from (4.2) is connected with the energy density of the soliton wave

$$H = \frac{1}{2}\left(\dot{v}^2 + v'^{\,2}\right) + 1 - \cos v \, .$$

The aim of this subsection is to investigate some basic properties of the locally anisotropic sine-Gordon equation, which describes a 2D horizontal soliton-dilaton system (see the next section) induced anholonomically via a source Υ_3 in the vertical subspace.

4.1.1 Lorentzian anisotropic soliton systems of class 1

We consider the class 1 of lorentzian h-metrics (3.11) which in local coordinates $x^i = (t, r)$, t is a time like coordinate, and for $\epsilon = -1$ and $\Upsilon_3 = p_3 + (1/2\kappa)\lambda$, where p_3 is the anisotropic pressure in the z-direction and λ is the cosmological constant, are defined by the equation

$$(\ddot{v} + v'') + \frac{\cos v - 1}{\sin v}\left(\dot{v}^2 + v'^{\,2}\right) = -\left(\kappa p_3 + \frac{\lambda}{2}\right)\sin v \, , \tag{4.4}$$

where $\dot{v} = \partial v/\partial t$ and $v' = \partial v/\partial r$.

For $\cos v \simeq 1$, by neglecting quadratic terms v^2, we can approximate the solution of (4.4) by a solution of the euclidean 2D sine-Gordon equation (1.2) with the constant $\widetilde{m}^2 = -(\kappa p_3 + \lambda/2)$. If we wont to treat $\widetilde{m}^2$ as a mass like constant, we must suppose that matter is in a state for which $(\kappa p_3 + \lambda/2) < 0$.

4.1.2 Euclidean anisotropic soliton systems of class 2

This type of h-metrics (3.14), of class 2 from the previous section, for $\epsilon = 1$, $\Upsilon_3 = p_3 + (1/2\kappa)\lambda$ and both space like local coordinates x^i, is given by the following from (4.3) equations

$$(\ddot{v} - v'') + \frac{\cos v - 1}{\sin v}\left(\dot{v}^2 - v'^{\,2}\right) = \left(\kappa p_3 + \frac{\lambda}{2}\right)\sin v \, , \tag{4.5}$$

where $\dot{v} = \partial v/\partial x^1$ and $v' = \partial v/\partial x^2$, which for $\cos v \simeq 1$ has solutions approximated by the lorentzian 2D sine-Gordon equation with $\widetilde{m}^2 = (\kappa p_3 + \lambda/2)$.

4.1.3 Euclidean anisotropic soliton systems of class 3

In this case the h-metrics (3.17) of class 3 from the previous section, for $\epsilon = 1$, $\Upsilon_3 = -\varepsilon + (1/2\kappa)\lambda$ and both space like local coordinates x^i, is defined by equations

$$(\ddot{v} - v'') + \frac{\cos v - 1}{\sin v}\left(\dot{v}^2 - v'^{\,2}\right) = \left(-\kappa\varepsilon + \frac{\lambda}{2}\right)\, \sin v \, , \tag{4.6}$$

where $\dot{v} = \partial v/\partial x^1$ and $v' = \partial v/\partial x^2$, which for $\cos v \simeq 1$ has solutions approximated by the lorentzian 2D sine-Gordon equation with $\widetilde{m}^2 = (-\kappa\epsilon + \lambda/2)$.

4.1.4 A static one dimensional exact solution

The deformed sine-Gordon equations can be integrated exactly for static configurations. If $v = v_s(x^2 = x), v' = dv_s/dx$ the equation (4.3) transforms into

$$\frac{d^2 v_s}{dx^2} + \frac{\cos v_s - 1}{\sin v_s}\left(\frac{dv_s}{dx}\right)^2 + \kappa\Upsilon_3 \ \sin v_s = 0\,. \tag{4.7}$$

which does not depend on values of $\epsilon = \pm 1$. Introducing a new variable $y(v_s) = (dv_s/dx)^2$ we get a linear first order differential equation

$$\frac{dy}{dv_s} + 2\frac{\cos v_s - 1}{\sin v_s} y + 2\kappa\Upsilon_3 \ \sin v_s = 0$$

which is solved by applying standard methods [18].

4.2 Vertical Einstein equations and possible soliton like solutions

The basic equation

$$\frac{\partial^2 h_4}{\partial z^2} - \frac{1}{2h_4}\left(\frac{\partial h_4}{\partial z}\right)^2 - \frac{1}{2h_3}\left(\frac{\partial h_3}{\partial z}\right)\left(\frac{\partial h_4}{\partial z}\right) - 2\kappa\Upsilon_1 h_3 h_4 = 0 \tag{4.8}$$

(here we write down the partial derivatives on z in explicit form) follows from (3.3) and (3.8) and relates some first and second order partial on z derivatives of diagonal components $h_a(x^i, z)$ of a v-metric with a source $\kappa\Upsilon_1(x^i, z) = \kappa\Upsilon_1^1 = \kappa\Upsilon_2^2$ in the h-subspace. We can consider as unknown the function $h_3(x^i, z)$ (or, inversely, $h_4(x^i, z)$) for some compatible values of $h_4(x^i, z)$ (or $h_3(x^i, z)$) and source $\Upsilon_1(x^i, z)$.

The structure of equation (4.8) differs substantially from the horizontal one (3.7), or (4.3). In this Subsection we analyze some soliton type integral varieties which solve the partial differential equation (4.8).

4.2.1 Belinski-Zakharov-Maison locally isotropic limits

In the vacuum case $\Upsilon_1^1 \equiv 0$ and arbitrary two functions depending only on variables x^i are admitted as solutions of (4.8). The N-coefficients q_i and n_i became zero in consequence of (3.9), (3.10) and (3.4), (3.6). So, in the locally isotropic vacuum limit the 4D metrics (1.5) will transform into a soliton vacuum solution of Einstein equations if, for instance, we require that $h_a(x^i)$ are the components of the diagonalized matrix for the Belinski-Zakharov [4] or Maison [24] gravitational solitons and the h-metric transforms is defined by a conformal factor $f(x^i)$ being compatible with the v-metric. Of course, instead of soliton ones we can choose another class of vacuum solutions depending on variables x^i to be the locally isotropic limit of anholonomic gravitational systems.

4.2.2 Kadomtsev-Petviashvili v-solitons

By straightforward verification we conclude that the v-metric component $h_4(x^i, z)$ could be a solution of Kadomtsev-Petviashvili (KdP) equation [17] (the first methods of integration of 2+1 dimensional soliton equations where developed by Dryuma [7] and Zakharov and Shabat [36])

$$h_4^{**} + \epsilon \left(\dot{h}_4 + 6h_4 h_4' + h_4''' \right)' = 0, \epsilon = \pm 1 \,, \tag{4.9}$$

if the component $h_3(x^i, z)$ satisfies the Bernoulli equations [18]

$$h_3^* + Y\left(x^i, z\right)(h_3)^2 + F_\epsilon\left(x^i, z\right) h_3 = 0 \,, \tag{4.10}$$

where, for $h_4^* \neq 0$,

$$Y\left(x^i, z\right) = 4\kappa \Upsilon_1^1 \frac{h_4}{h_4^*} \,, \tag{4.11}$$

and

$$F_\epsilon\left(x^i, z\right) = \frac{h_4^*}{h_4} + \frac{2\epsilon}{h_4^*} \left(\dot{h}_4 + 6h_4 h_4' + h_4''' \right)' .$$

The three dimensional integral variety of (4.10) is defined by formulas

$$h_3^{-1}\left(x^i, z\right) = h_{3(x)}^{-1}\left(x^i\right) E_\epsilon\left(x^i, z\right) \times \int \frac{Y\left(x^i, z\right)}{E_\epsilon\left(x^i, z\right)} dz \,,$$

where

$$E_\epsilon\left(x^i, z\right) = \exp \int F_\epsilon\left(x^i, z\right) dz$$

and $h_{3(x)}\left(x^i\right)$ is a nonvanishing function.

In the vacuum case $Y\left(x^i, z\right) = 0$ and we can write the integral variety of (4.10)

$$h_3^{(vac)}\left(x^i, z\right) = h_{3(x)}^{(vac)}\left(x^i\right) \exp\left[- \int F_\epsilon\left(x^i, z\right) dz \right] .$$

We conclude that a solution of KdP equation (4.10) could generate a non-perturbative component $h_4(x^i, z)$ of a diagonal h-metric if the second component $h_3\left(x^i, z\right)$ is a solution of Bernoulli equations (4.10) with coefficients determined both by h_4 and its partial derivatives and by the Υ_1^1 component of the energy-momentum d-tensor (see (4.11)). In the non-vacuum case the parameters of (2+1) dimensional KdV solitons are connected with parameters defining the interactions with matter fields and/or by a cosmological constant. The further developments in this direction consist in construction of self-consistent (2+1) KdV soliton solutions induced by some soliton configurations from energy-momentum tensor in hydrodynamical (plasma) approximations.

4.2.3 (2+1) sine-Gordon v-solitons

In a symilar manner as in previous paragraph we can prove that solutions $h_4(x^i, z)$ of (2+1) sine-Gordon equation (see, for instance, [12, 23, 35])

$$h_4^{**} + h_4^{''} - \ddot{h}_4 = \sin(h_4)$$

also induce solutions for $h_3\left(x^i, z\right)$ following from the Bernoulli equation

$$h_3^* + 4\kappa\Upsilon_1(x^i, z)\frac{h_4}{h_4^*}(h_3)^2 + F\left(x^i, z\right) h_3 = 0, h_4^* \neq 0\,,$$

where

$$F\left(x^i, z\right) = \frac{h_4^*}{h_4} + \frac{2}{h_4^*}\left[h_4^{''} - \ddot{h}_4 - \sin(h_4)\right].$$

The integral varieties (with energy-momentum sources and in vacuum cases) are constructed by a corresponding redefinition of coefficients in formulas from the previous paragraph.

4.2.4 On some general properties of h-metrics depending on 2+1 variables

By introducing a new variable $\beta = h_4^*/h_4$ the equation (4.8) transforms into

$$\beta^* + \frac{1}{2}\beta^2 - \frac{\beta h_3^*}{2h_3} - 8\kappa\Upsilon_1 h_3 = 0 \qquad (4.12)$$

which relates two functions $\beta\left(x^i, z\right)$ and $h_3\left(x^i, z\right)$. There are two possibilities: 1) to define β (i.e. h_4) when $\kappa\Upsilon_1$ and h_3 are prescribed and, inversely 2) to find h_3 for given $\kappa\Upsilon_1$ and h_4 (i.e. β); in both cases one considers only "*" derivatives on z-variable with coordinates x^i treated as parameters.

1. In the first case the explicit solutions of (4.12) have to be constructed by using the integral varieties of the general Riccati equation [18] which by a corresponding redefinition of variables, $z \to z\left(\varsigma\right)$ and $\beta\left(z\right) \to \eta\left(\varsigma\right)$ (for simplicity, we omit dependencies on x^i) could be written in the canonical form

$$\frac{\partial\eta}{\partial\varsigma} + \eta^2 + \Psi\left(\varsigma\right) = 0\,,$$

where Ψ vanishes for vacuum gravitational fields. In vacuum cases the Riccati equation reduces to a Bernoulli equation which (we can use the former variables) for $s(z) = \beta^{-1}$ transforms into a linear differential (on z) equation,

$$s^* + \frac{h_3^*}{2h_3}s - \frac{1}{2} = 0\,. \qquad (4.13)$$

2. In the second (inverse) case when h_3 is to be found for some prescribed $\kappa\Upsilon_1$ and β the equation (4.12) is to be treated as a Bernoulli type equation,

$$h_3^* = -\frac{4\kappa\Upsilon_1}{\beta}(h_3)^2 + \left(\frac{2\beta^*}{\beta} + \beta\right) h_3 \qquad (4.14)$$

which can be solved by standard methods. In the vacuum case the squared on h_3 term vanishes and we obtain a linear differential (on z) equation.

4.2.5 A class of conformally equivalent h-metrics

A particular interest presents those solutions of the equation (4.12) which via 2D conformal transforms with a factor $\omega = \omega(x^i, z)$ are equivalent to a diagonal h-metric on x-variables, i.e. one holds the parametrization

$$h_3 = \omega(x^i, z)\ a_3\left(x^i\right) \qquad \text{and} \qquad h_4 = \omega(x^i, z)\ a_4\left(x^i\right), \qquad (4.15)$$

where $a_3\left(x^i\right)$ and $a_4\left(x^i\right)$ are some arbitrary functions (for instance, we can impose the condition that they describe some 2D soliton like or black hole solutions). In this case $\beta = \omega^*/\omega$ and for $\gamma = \omega^{-1}$ the equation (4.12) transforms into

$$\gamma\ \gamma^{**} = -8\kappa\Upsilon_1 a_3\left(x^i\right) \qquad (4.16)$$

with the integral variety determined by

$$z = \int \frac{d\gamma}{\sqrt{|-16k\Upsilon_1 a_3(x^i)\ln|\gamma| + C_1(x^i)|}} + C_2(x^i)\,, \qquad (4.17)$$

where it is considered that the source Υ_1 does not depend on z.

Finally, in this section, we have shown that a large class of 4D solutions, depending on two or three variables, of the Einstein equations can be constructed as nonlinear superpositions of some 2D h-metrics defined by locally anisotropic deformations of 2D sine-Gordon equations and of some v-metrics generated in particular by solutions of Kadomtsev-Petviashvili equations, or of (2+1) sine-Gordon, and associated Bernoulli type equations. From a general viewpoint the v-metrics are defined by integral varieties of corresponding Riccati and/or Bernoulli equations with respect to z-variables with the h-coordinates x^i treated as parameters.

5. Effective locally anisotropic soliton-dilaton fields

The formula for the h-component $\widehat{R}$ of scalar curvature (see (2.7) and (3.2)) of a h-metric (1.6), written for a anisotropic system, differs from the usual one for computation of curvature of 2D metrics. That why additionally to the first term in (4.1) it is induced the ρ-term (4.2). The aim of this section is to prove that the ansiotropically deformed singe-Gordon equation (4.1) could be equivalently modelled by solutions of the usual 2D singe-Gordon equation and an additional equation for a corresponding effective dilaton field (in brief, by a soliton-dilaton field). We also analyze the 2D dilaton gravity in connection with the sine-Gordon anisotropic field theory.

5.1 Generic locally anisotropic dilaton fields

Let $\widetilde{g}^{\epsilon}_{ij}\left(x^i\right)$ be a 2D metric of Lorentz (or euclidean) signature for $\epsilon = -1$ (or $\epsilon = 1$) with a usual 2D scalar curvature $\widetilde{R}_{(\epsilon)}\left(x^i\right)$. We also consider a conformally equivalent metric

$$\underline{g}^{\epsilon}_{ij}\left(x^i\right) = \exp\omega\left(x^i\right)\widetilde{g}^{\epsilon}_{ij}\left(x^i\right). \tag{5.1}$$

The scalar curvatures of 2D metrics from (5.1) are related by the formula

$$e^{\omega}\underline{R} = \widetilde{R}_{(\epsilon)} + \triangle_{(\epsilon)}\omega\,, \tag{5.2}$$

where $\triangle_{(\epsilon)}$ is the d'Alambert, $\epsilon = -1$, (Laplace, $\epsilon = 1$) operator.

In order to model a locally anisotropic 2D horizontal system via a locally isotropic 2D gravity we consider that

$$\widehat{R} = e^{\omega}\underline{R}, \widetilde{R}_{(\epsilon)} = -2\widetilde{m}^2$$

and

$$\triangle_{(\epsilon)}\omega = \rho\left(x^i\right). \tag{5.3}$$

For a given 'tilded' metric, for instance, $\widetilde{g}_{ij} = \mathrm{diag}(\widetilde{a},\widetilde{b})$ being a solution of 2D sine-Gordon equation (1.2) with negative constant scalar curvature (see (1.3)), the wave (Poisson) equation can be solved in explicit form by imposing corresponding boundary conditions.

So, a 2D locally anisotropic h-space is equivalently modelled by a usual curved 2D locally isotropic (pseudo) riemannian space and effective interactions with the generic locally anisotropic dilaton field $\Phi_{(\omega)} = \exp\omega$.

5.2 Locally anisotropic 2D dilaton gravity and sine-Gordon theory

In the previous Subsection the conformal factor $\Phi_{(\omega)}$, in the h-space, was introduced with the aim to compensate the local anisotropy, induced from the v-space. The 2D h-gravity can be formulated as a dilatonic one related to a generalized, anisotropically deformed, sine-Gordon model.

By using Weyl rescallings of h-metric (1.6) one can write the general action for, in our case, the h-model (see the isotropic variant in [2] and [25]),

$$S^{[h]}\left[g_{ij},\Phi\right] = \frac{1}{2\pi}\int d^2x\sqrt{-g}[\Phi\widehat{R} + \varpi^2 V\left(\Phi\right)], \tag{5.4}$$

where the h-metric g_{ij} has signature $(-1,1)$, $V(\Phi)$ is an arbitrary function of the dilaton field Φ and ϖ is the connection constant. The anisotropic field equations derived from this action are

$$\widehat{R} = \widetilde{R}_{(-)} + \triangle_{(-)}\omega = -\varpi^2 \frac{dV}{d\Phi}\,, \tag{5.5}$$

$$D_i D_j \Phi - \frac{\varpi^2}{2} g_{ij} V = 0\,,$$

where $\widetilde{R}_{(-)} = 2\widetilde{R}_1^1$ is defined by the h-component of scalar curvature of type (4.1), when $\epsilon = -1$. In the locally isotropic limit this system of equations describes the Cadoni theory [6].

In consequence of the fact that the theory is invariant under coordinate h-transforms $(x^1 = t, x^2 = x)$ we can introduce the h-metric

$$g^{[h]} = -\sin^2\left(\frac{v}{2}\right) dt^2 + \cos^2\left(\frac{v}{2}\right) dx^2\,, \tag{5.6}$$

where $v = v(t,x)$ and rewrite the system (5.5) as a system of nonlinear partial differential equations in 2D euclidean space,

$$\ddot{v} + v'' = \left(-\rho + \frac{\varpi^2}{2}\frac{dV}{d\Phi}\right)\sin v\,, \tag{5.7}$$

$$\ddot{\Phi} + \Phi'' = \frac{\varpi^2}{2} V \cos v\,, \tag{5.8}$$

where the function ρ is defined by the formula (4.2) for $\epsilon = -1$, or, in equivalent form, by a generic anisotropic dilaton given by (5.3).

The equation (5.7) reduces to the deformed sine-Gordon equation (4.3), for $\epsilon = -1$, if $V = \Phi$, or for constant configurations Φ_0 for which $V(\Phi_0) = 0$ and $dV/d\Phi|_{\Phi_0} > 0$.

For soliton-dilaton anisotropic configurations it is more convenient to consider the action

$$S = \frac{1}{2}\int d^2x \left[\Phi\left(\triangle_{(-)}v + \rho\right) - \frac{\varpi^2}{2} V \sin v\right], \tag{5.9}$$

given in the 2D Minkowski space, where $\triangle_{(-)}v = \ddot{v} + v''$ and $\rho = \triangle_{(-)}\omega$. Extremizing the action (5.9) we obtain the field equations (5.7) and (5.8) as well from this action one follows the energy functional

$$E(v,\omega,\Phi) = \frac{1}{2}\int\limits_{-\infty}^{\infty} dx \left[\dot{\Phi}\left(\dot{v} + \dot{\omega}\right) + \Phi'\left(v' + \omega'\right) + \frac{\varpi^2}{2} V \sin v\right]. \tag{5.10}$$

We note that instead of Lorentz type 2D h-metrics we can consider euclidean field equations by performing the Wick rotation $t \to it$. We conclude this Subsection by the remark that a complete correspondence between locally anisotropic h-metrics and dilaton structures is possible if additionally to trigonometric parametrizations of 2D metrics (5.6) one introduces hyperbolic parametizations

$$g^{[h]} = -\sinh^2\left(\frac{v}{2}\right) dt^2 + \cosh^2\left(\frac{v}{2}\right) dx^2$$

which results in sinh-Gordon models.

5.3 Static locally anisotropic soliton-dilaton configurations

Because the ρ-term (4.2) vanishes for constant values of fields $v = 2n\pi$, $n = 0, \pm 1, \pm 2, \ldots$ and $\Phi = \Phi_0$ the vacua of the model (5.4) is singled out like in the locally isotropic case [6], by conditions

$$V(\Phi_0) = 0 \text{ and } \frac{dV}{d\Phi}(\Phi_0) > 0\,. \tag{5.11}$$

In order to focus on static deformations induced by soliton like solutions we require $E \geq 0$ and

$$\lim_{x\to\pm\infty.} v' \to 0 \qquad \text{and} \qquad \lim_{x\to\pm\infty.} \Phi' \to 0\,. \tag{5.12}$$

The static anisotropic configurations of (5.7) and (5.8) are giving by anholonomic deformation of isotropic ones and are given by the system of equations

$$v'' + \frac{\cos v - 1}{\sin v}(v')^2 = \frac{\varpi^2}{2}\sin v\,\frac{dV}{d\Phi}\,, \tag{5.13}$$

$$\Phi'' = \frac{\varpi^2}{2} V(\Phi)\cos v\,. \tag{5.14}$$

The first integrals of (5.13) and (5.14) are

$$v' = \varpi\frac{a_0}{\sqrt{2}}\sin^2\frac{v}{2} \text{ and } \Phi' = \frac{\varpi\sqrt{2}}{a_0}\cot\frac{v}{2}$$

which, after another integration, results in the solutions

$$\varpi(x - x_0) = \pm\frac{a_0^2}{\sqrt{2}}\int d\Phi\sqrt{\frac{\Psi - c_0}{1 - a_0^2(\Psi - c_0)}}\,,$$

$$\sin\frac{v}{2} = \pm a_0\sqrt{\Psi - c_0}\,, \tag{5.15}$$

where $\Psi = \Psi(\Phi) = \int_0^{\Phi} d\tau V(\tau)$; a_0 and $c_0 = \Psi[\underline{\Phi}(\pm\infty)]$ are integration constants. We emphasize that the formula (5.15), following from an anisotropic model, differs from that obtained in the Cadoni's locally isotropic theory [6].

There are two additional two parameter solutions of (5.13) and (5.13) which are not contained in (5.15). The first type of solutions are those for constant v field when

$$v = n\pi \text{ and } \varpi(x - x_0) = \pm\int d\Phi\left[(-1)^n\Psi - b_0\right]^{-1/2}\,,$$

where $b_0 = \text{const}$. The second type of solutions are for constant dilaton fields Φ_0 and exists if there is at least one zero $\Phi = \Phi_0$ for $V(\Phi)$. For $dV/d\Phi(\Phi_0) > 0$ the equations reduce to the usual sine-Gordon equations

$$v'' = \frac{\varpi^2}{2}\frac{dV}{d\Phi}\,|_{\Phi_0}\,\sin v\,.$$

Note that the model (5.9) admits static soliton solutions, approaching for $x \to \pm\infty$ the constant field configuration $v = 2\pi n;\ n = \pm 1$, with $\Phi_0 = \Phi(\pm\infty)$, $V(\Phi_0) = 0$ and $dV/d\Phi|_{\Phi_0} > 0$.

5.4 Topology of locally anisotropic soliton-dilatons

If we suppose that anisotropic deformations do not change the spacetime topology, the conditions (5.12) imply that every soliton solution tends asymptotically to one of vacuum configurations (5.11) which could be considered for both locally anisotropic and isotropic systems (see [6]). The admissible number of solitons to be anisotropically deformed without changing of topology is determined by the number of ways in which the points $x = \pm\infty$ (the zero sphere) can be mapped into the manifold of the mentioned constant-field configurations (5.11) characterized by the homotopy group

$$\pi_0\left(\frac{Z\times Z_2}{Z_2}\right) = \pi_0(Z) = Z\,,$$

when $G = Z\times Z_2$ is the invariance group for a generic V, and Z and Z_2 are respectively the infinite discrete group translations and the finite group of inversions of the field v, parametrized by $v \to v + 2\pi n$, $v \to -v$. This result holds for usual sine-Gordon systems, as well by anisotropic generalizations given by the action (5.9) and anisotropic field equations (5.7) and (5.8).

For soliton like theories it is possible the definition of conserved currents

$$J^i_{(v)} = \epsilon^{ij}\delta_i v \text{ and } J^i_{(\Phi)} = \epsilon^{ij}\delta_i\Phi\,,$$

where $\epsilon^{ij} = -\epsilon^{ji}$ and the 'elongated' (in the anisotropic case) partial derivatives δ_i are given by (1.9). The associated topological charges on a fixed anisotropic background are

$$Q_{(v)} = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} dx\,J^1_{(v)} = \frac{1}{2\pi}\left[v(\infty) - v(-\infty)\right],$$

$$Q_{(\Phi)} = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} dx\,J^1_{(\Phi)} = \frac{1}{2\pi}\left[\Phi(\infty) - \Phi(-\infty)\right].$$

The topological properties of anisotropic backgrounds are characterized by the topological current and charge of anisotropic dilaton $\Phi_{(\omega)} = \exp\omega$ defined by a solution of Poisson equation (5.3). The corresponding formulas are

$$J^i_{(e^\omega)} = \epsilon^{ij}\delta_i e^\omega \,,$$
$$Q_{(e^\omega)} = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} dx J^1_{(e^\omega)} = \frac{1}{2\pi}\left[\exp\omega\left(\infty\right) - \exp\omega\left(-\infty\right)\right]. \tag{5.16}$$

If $J^i_{(e^\omega)}$ and $Q_{(e^\omega)}$ are non-trivial we can conclude that our soliton-dilaton system was topologically changed under anisotropic deformations.

6. Locally anisotropic black holes and solitons

In this section we analyze the connection between h-metrics describing effective 2D black hole anisotropic solutions (with parameters defined by v-components of a diagonal 4D energy-momentum d-tensor) and 2D soliton anisotropic solutions obtained in the previous two sections.

6.1 A Class 1 black hole solutions

Let us consider a static h-metric of type (3.11), for which $g_1 = -\alpha(r)$ and $g_2 = 1/\alpha(r)$ for a function on necessary smooth class α on h-coordinates $x^1 = T$ and $x^2 = r$. Putting these values of h-metric into (3.2) we compute

$$R^1_1 = R^2_2 = -\frac{1}{2}\alpha'' \,.$$

Considering the 2D h-subspace to be of constant negative scalar curvature,

$$\widehat{R} = 2R^1_1 = -\widetilde{m}^2 \,,$$

and that the Einstein anisotropic equations (3.7) are satisfied we obtain the relation

$$\alpha'' = \widetilde{m}^2 = \kappa\Upsilon^3_3 = \kappa\Upsilon^4_4 \,, \tag{6.1}$$

which for a diagonal energy-momentum d-tensor with $\kappa\Upsilon^3_3 = kp_3 + \lambda/2$ transforms into

$$\widetilde{m}^2 = kp_3 + \frac{\lambda}{2} \,.$$

The solution of (6.1) is written in the form $\alpha = (\widetilde{m}^2 r^2 - M)$ which defines a 2D h-metric

$$ds^2_{(h)} = -\left(\widetilde{m}^2 r^2 - M\right) dT^2 + \left(\widetilde{m}^2 r^2 - M\right)^{-1} dr^2 \tag{6.2}$$

being similar to a black hole solution in 2D Jackiw-Teitelboim gravity [16] and display many of attributes of black holes [25, 11, 22] with that difference that the constant $\widetilde{m}$ is defined by 4D physical values in v-subspace and for definiteness of the theory the h-metric should be supplied with a v-component.

The parameter M, the mass observable, is the analogue of the Arnowitt-Deser-Misner (ADM) mass in general relativity. If we associate the h-metric (6.2) to an anisotropic model of 2D model of Jackiw-Teitelboim gravity, given by the action

$$I_{JT}[\phi, g] = \frac{1}{2G_{(2)}} \int_H d^2x \sqrt{|g|}\phi \left(\widehat{R} + 2\widetilde{m}^2\right),$$

where $\widehat{R}$ is the h-component of the Ricci d-curvature, ϕ is the dilaton field and $G_{(2)}$ is the gravitational coupling constant in 2D, we should add to (6.1) the field equation for ϕ,

$$\left(D_i D_j - \widetilde{m}^2 g_{ij}\right)\phi = 0$$

which has the solution

$$\phi = c_1 \widetilde{m} r$$

with coupling constant c_1, (we can consider $c_1 = 1$ for the vacuum configurations $\phi = \widetilde{m} r$ as $r \to \infty$). In this case the mass observable is connected with the dilaton as

$$M = -\widetilde{m}^{-2} \left|D\phi\right|^2 + \phi^2 . \tag{6.3}$$

Clearly this model is with local anisotropy because the value $\widehat{R}$ is defined in an anisotropic manner and not as a usual scalar curvature in 2D gravity.

6.2 Deformed soliton-dilaton systems and black holes with anisotropy

Suppose we have a h-metric (5.6) which must solve the equations

$$\ddot{v} + v'' = \left(-\rho + \widetilde{m}^2\right)\sin v , \tag{6.4}$$

$$\ddot{\phi} + \phi'' = \widetilde{m}^2 \cos v . \tag{6.5}$$

These equations are a particular case of the system (5.7) and (5.8). For $\rho = 0$ such equations were investigated in [10]. In the previous section we concluded that every 2D anisotropic system can be equivalently modelled in an isotropic space by considering an effective interaction with an anisotropic dilaton field. The same considerations hold good for 2D anisotropic spaces with that remark that the dilaton field ϕ must be composed from a component satisfying the equation (6.5) and another component defined from the Poisson equation (5.3).

Having a dilaton field $\phi(t, x)$ we can introduce a new "radial coordinate"

$$r(t, x) := \phi / \widetilde{m}$$

which (being substituted into h-metric (5.6)) results in the horizontal 2D metric

$$ds^2_{[h]} = -\widetilde{m}^{-2} \left|D\phi\right|^2 dT^2 + \widetilde{m}^2 \left|D\phi\right|^{-2} dr^2 , \tag{6.6}$$

where

$$dT \doteq \left|D\phi\right|^{-2} \left(\phi' \tan\frac{v}{2}\, dt + \dot{\phi} \cot\frac{v}{2}\, dx\right).$$

The metric (6.6) is the same as (6.2) because the mass observable is defined by (6.3).

6.3 The geometry of black holes with anisotropy and deformed one-soliton solutions

The one soliton solution of the euclidean sine-Gordon equation can be written as

$$v(t,x) = 4\tan^{-1}\left\{\exp\left[\pm\widetilde{m}\gamma\left(x - st - x_{(0)}\right)\right]\right\}, \tag{6.7}$$

where $\gamma = 1/\sqrt{1+s^2}$, s is the spectral parameter and the integration constant $x_{(0)}$ is considered, for simplicity, zero. The solution with signs $+/-$ gives a soliton/anti-soliton configuration. Let us demonstrate that a corresponding black hole with anisotropy can be constructed. Putting (6.7) into the h-metric (5.6) we obtain a lorentzian one-soliton 2D metric

$$ds^2_{[1-sol]} = -\sec h^2\xi\ dt^2 + \tanh^2\xi\ dx^2\,,$$

where

$$\xi \doteq \widetilde{m}\gamma\,(x - st)\,.$$

In a similar fashion we can compute (by using the function (6.7)) the anisotropic deformation ρ (4.2) and effective anisotropic dilaton $\Phi_\omega = \exp\omega$ which follow from the Poisson equation (5.3). In both cases of dilaton equations we are dealing with linear partial differential equations. A combination of type

$$\phi = \phi_{[0]}\dot{v} + \phi_{[1]}v' \tag{6.8}$$

for arbitrary constants $\phi_{[0]}$ and $\phi_{[1]}$ satisfies the linearized sine-Gordon equation and because for the function (6.7)

$$\dot{v} = \mp 4\widetilde{m}\gamma s\ \sec h\ \xi = -sv'$$

we can put in (6.8) $\phi_{[1]} = 0$ and following a hamiltonian analysis (in order to have compatibility with the locally isotropic case [10]; for a black hole mass $M = s^2$ with corresponding ADM energy $E = \widetilde{m}^2 s^2/\left(2G_{(2)}\right))$ we set $\phi_{[0]} = 1/\left(4\widetilde{m}\gamma^2\right)$ so that

$$\phi = \left|\frac{s}{\gamma}\right| \sec h\ \xi$$

is chosen to make ϕ positive. In consequence, the black hole coordinates (r,T) (anisotropic deformations reduces to reparametrization of such coordinates) are defined by

$$r = \phi/\widetilde{m} = \frac{s}{\widetilde{m}\gamma}\sec h\ \xi$$

and

$$dT = |s\widetilde{m}|^{-1}\left[dt - \frac{s\tanh^2\ \xi}{\widetilde{m}\gamma\left(\sec h^2\ \xi - s^2\tanh^2\ \xi\right)}d\xi\right].$$

With respect to these coordinates the obtained black hole metric is of the form

$$ds^2_{[bh]} = -\left(\widetilde{m}^2 r^2 - \widetilde{m}^2 s^4\right) dT^2 + \left(\widetilde{m}^2 r^2 - \widetilde{m}^2 s^4\right)^{-1} dr^2$$

which describes a Jackiw-Teitelboim black hole with mass parameter (6.3)

$$M_{1sol} = \widetilde{m}^2 s^4 \,,$$

defined by the corresponding component $\Upsilon_3^3 = \Upsilon_4^4$ of energy-momentum d-tensor in v-space and spectral parameter s of the one soliton background, and event horizon at $\phi = \phi_H = s^2$.

In a similar fashion we can use instead of the function (6.7) a two and, even multi-soliton background. The calculus is similar to the locally isotropic case [10], having some redefinitions of black hole coordinates if it is considered that anisotropic deformations do not change the h-spaces topology, i.e the anisotropic gravitational topological source and charge (5.16) vanishes.

7. 3D black holes with anisotropy

Let us analyze some basic properties of 3D spacetime $V^{(2+1)}$ provided with d-metrics of type

$$\delta s^2 = g_1\left(x^k\right)\left(dx^1\right)^2 + g_2\left(x^k\right)\left(dx^2\right)^2 + h_3(x^i,z)\left(\delta z\right)^2 \,, \tag{7.1}$$

where x^k are 2D coordinates, $y^3 = z$ is the anisotropic coordinate and

$$\delta z = dz + N_i^3(x^k,z)dx^i \,.$$

The N-connection coefficients are given by some functions on x^i and z,

$$N_1^3 = q_1(x^i,z),\ N_2^3 = q_2(x^i,z)\,. \tag{7.2}$$

The non-trivial components of the Ricci d-tensor (2.6) are

$$R_1^1 = R_2^2 = \frac{1}{2g_1g_2}\left[-(g_1^{''} + \ddot{g}_2) + \frac{1}{2g_2}\left(\dot{g}_2^2 + g_1'g_2'\right) + \frac{1}{2g_1}\left(g_1'^{\,2} + \dot{g}_1\dot{g}_2\right)\right] \tag{7.3}$$

(for 3D the component $S_3^3 \equiv 0$, see (3.3)).

The curvature scalar $\overleftarrow{R}$ (2.7) is $\overleftarrow{R} = \widehat{R} = 2R_1^1$.

The system of Einstein equations (2.8) transforms into

$$R_1^1 = -\kappa\Upsilon_3^3 \,, \tag{7.4}$$

which is compatible for if the 3D matter is in a state when for the energy-momentum d-tensor Υ_β^α one holds $\Upsilon_1^1 = \Upsilon_2^2 = 0$ and the value of R_1^1 is taken from (7.3).

7.1 Static elliptic horizons

Let us consider a class of 3D d-metrics which local anisotropy which are similar to Banados-Teitelboim-Zanelli (BTZ) black holes [1].

The d-metric is parametrized

$$\delta s^2 = g_1\left(\chi^1,\chi^2\right)(d\chi^1)^2 + \left(d\chi^2\right)^2 - h_3\left(\chi^1,\chi^2,t\right)\ (\delta t)^2\ , \tag{7.5}$$

where $\chi^1 = r/r_h$ for $r_h = \text{const}$, $\chi^2 = \theta/r_a$ if $r_a = \sqrt{|\kappa\Upsilon_3^3|} \neq 0$ and $\chi^2 = \theta$ if $\Upsilon_3^3 = 0$, $y^3 = z = t$, where t is the time like coordinate. The Einstein equations (7.4) transform into

$$\frac{\partial^2 g_1}{\partial(\chi^2)^2} - \frac{1}{2g_1}\left(\frac{\partial g_1}{\partial\chi^2}\right)^2 - 2\kappa\Upsilon_3^3 g_1 = 0\,. \tag{7.6}$$

By introducing new variables

$$p = \frac{g_1'}{g_1}\,, \tag{7.7}$$

where the 'prime' in this subsection denotes the partial derivative ∂/χ^2, the equation (7.6) transform into

$$p' + \frac{p^2}{2} + 2\epsilon = 0\,. \tag{7.8}$$

The integral curve of (7.8), intersecting a point $\left(\chi^2_{(0)}, p_{(0)}\right)$, considered as a differential equation on χ^2 is defined by the functions [18]

$$p = \frac{p_{(0)}}{1+\frac{p_{(0)}}{2}\left(\chi^2-\chi^2_{(0)}\right)}, \qquad \epsilon = 0\,; \tag{7.9}$$

$$p = \frac{p_{(0)} - 2\tanh\left(\chi^2-\chi^2_{(0)}\right)}{1+\frac{p_{(0)}}{2}\tanh\left(\chi^2-\chi^2_{(0)}\right)}\,, \qquad \epsilon > 0\,; \tag{7.10}$$

$$p = \frac{p_{(0)} - 2\tan\left(\chi^2-\chi^2_{(0)}\right)}{1+\frac{p_{(0)}}{2}\tan\left(\chi^2-\chi^2_{(0)}\right)}, \qquad \epsilon < 0\,. \tag{7.11}$$

Because the function p depends also parametrically on variable χ^1 we must consider functions $\chi^2_{(0)} = \chi^2_{(0)}\left(\chi^1\right)$ and $p_{(0)} = p_{(0)}\left(\chi^1\right)$.

For simplicity, here we elucidate the case $\epsilon < 0$. The general formula for the non-trivial component of h-metric is to be obtained after integration on χ^1 of (7.11) (see formula (7.7))

$$g_1\left(\chi^1,\chi^2\right) = g_{1(0)}\left(\chi^1\right)\left\{\sin[\chi^2 - \chi^2_{(0)}\left(\chi^1\right)] + \arctan\frac{2}{p_{(0)}\left(\chi^1\right)}\right\}^2\,,$$

for $p_{(0)}\left(\chi^1\right) \neq 0$, and

$$g_1\left(\chi^1,\chi^2\right) = g_{1(0)}\left(\chi^1\right)\ \cos^2[\chi^2 - \chi^2_{(0)}\left(\chi^1\right)] \tag{7.12}$$

for $p_{(0)}(\chi^1) = 0$, where $g_{1(0)}(\chi^1)$, $\chi^2_{(0)}(\chi^1)$ and $p_{(0)}(\chi^1)$ are some functions of necessary smoothness class on variable $\chi^1 = x^1/\sqrt{\kappa\varepsilon}$, when ε is the energy density. If we consider $\Upsilon_{3i} = 0$ and a non-trivial diagonal components of energy-momentum d-tensor, $\Upsilon^{\alpha}_{\beta} = \mathrm{diag}[0, 0, -\varepsilon]$, the N-connection coefficients $q_i(\chi^i, t)$ could be arbitrary functions.

For simplicity, in our further considerations we shall apply the solution (7.12).

The d-metric (7.5) with the coefficients (7.12) gives a general description of a class of solutions with generic local anisotropy of the Einstein equations (2.8).

Let us construct 3D static black hole anisotropic solutions.

We choose some coefficients $h_{3(0)}(\chi^k), g_{1(0)}(\chi^1)$ and $\chi_0(\chi^1)$ following physical considerations. For instance, the Schwarzschild solution is selected from a general 4D metric with some general coefficients of static, spherical symmetry by relating the radial component of metric with the Newton gravitational potential. In this section, we construct a locally anisotropic BTZ like solution by supposing that it is conformally equivalent to the BTZ solution if one neglects anisotropies on angle θ,

$$g_{1(0)}\left(\chi^1\right) = \left[r\left(-M_0 + \frac{r^2}{l^2}\right)\right]^{-2},$$

where $M_0 = \mathrm{const} > 0$ and $-1/l^2$ is a constant which is to be considered the cosmological from the locally isotropic limit. The time-time coefficient of d-metric is chosen

$$h_3\left(\chi^1, \chi^2\right) = r^{-2}\lambda_3\left(\chi^1, \chi^2\right)\cos^2[\chi^2 - \chi^2_{(0)}\left(\chi^1\right)]. \qquad (7.13)$$

If we chose in (7.13)

$$\lambda_3 = \left(-M_0 + \frac{r^2}{l^2}\right)^2,$$

when the constant

$$r_h = \sqrt{M_0}l$$

defines the radius of a circular horizon, the anisotropic solution is conformally equivalent, with the factor $r^{-2}\cos^2[\chi^2 - \chi^2_{(0)}(\chi^1)]$, to the BTZ solution embedded into a anholonomic background given by arbitrary functions $q_i(\chi^i, t)$ which are defined by some initial conditions of gravitational anisotropic background polarization.

A more general class of anisotropic solutions could be generated if we put, for instance,

$$\lambda_3\left(\chi^1, \chi^2\right) = \left(-M\left(\theta\right) + \frac{r^2}{l^2}\right)^2,$$

with

$$M\left(\theta\right) = \frac{M_0}{(1 + e\cos\theta)^2},$$

where $e < 1$. This solution has a horizon, $\lambda_3 = 0$, parametrized by an ellipse

$$r = \frac{r_h}{1 + e\cos\theta}$$

with parameter r_h and eccentricity e.

We note that our solution with elliptic horizon was constructed for a diagonal energy-momentum d-tensor with non-trivial energy density but without cosmological constant. On the other hand the BTZ solution was constructed for a generic 3D cosmological constant. There is not a contradiction here because the anisotropic solutions can be considered for a d-tensor $\Upsilon_{\beta}^{\alpha} = \mathrm{diag}[p_1 - 1/l^2, p_2 - 1/l^2, -\varepsilon - 1/l^2]$ with $p_{1,2} = 1/l^2$ and $\varepsilon_{(eff)} = \varepsilon + 1/l^2$ (for $\varepsilon = \mathrm{const}$ the last expression defines the effective constant r_a). The locally isotropic limit to the BTZ black hole could be realized after multiplication on r^2 and by approximations $e \simeq 0$, $\cos[\theta - \theta_0(\chi^1)] \simeq 1$ and $q_i(x^k, t) \simeq 0$.

7.2 Oscillating elliptic horizons

The simplest way to construct 3D solutions of the Einstein equations with oscillating in time horizon is to consider matter states with constant nonvanishing values of $\Upsilon_{31} = \mathrm{const}$. In this case the coefficient h_3 could depend on t-variable. For instance, we can chose such initial values when

$$h_3(\chi^1, \theta, t) = r^{-2}\left(-M(t) + \frac{r^2}{l^2}\right)\cos^2[\theta - \theta_0\left(\chi^1\right)] \qquad (7.14)$$

with

$$M = M_0 \exp(-\widetilde{p}t) \sin \widetilde{\omega} t\,,$$

or, for an another type of anisotropy,

$$h_3(\chi^1, \theta, t) = r^{-2}\left(-M_0 + \frac{r^2}{l^2}\right)\cos^2\theta\ \sin^2[\theta - \theta_0\left(\chi^1, t\right)] \qquad (7.15)$$

with

$$\cos\theta_0\left(\chi^1, t\right) = e^{-1}\left(\frac{r_a}{r}\cos\omega_1 t - 1\right),$$

when the horizon is given parametrically,

$$r = \frac{r_a}{1 + e\cos\theta}\cos\omega_1 t\,,$$

where the new constants (comparing with those from the previous subsection) are fixed by some initial and boundary conditions as to be $\widetilde{p} > 0$, and $\widetilde{\omega}$ and ω_1 are treated as some real numbers.

A solution (7.1) of the Einstein equations (7.4) with $g_2(\chi^i) = 1$ and $g_1(\chi^1, \theta)$ and $h_3(\chi^1, \theta, t)$ given respectively by formulas (7.12) and (7.14) describe a 3D evaporating black anisotropic hole solution with circular oscillating in time horizon. An another type of solution, with elliptic oscillating in time horizon, could be obtained if we choose (7.15).

8. 4D locally anisotropic black holes

8.1 Basic properties

The purpose of this section is the construction of d-metrics of Class 2, or 3 (see (3.14) and (3.15), or (3.17) and (3.18)) which are conformally equivalent to some anisotropic deformations of black hole, disk, torus and cylinder like solutions. We shall analyze 4D d-metrics of type

$$\delta s^2 = g_1\left(x^k\right)\left(dx^1\right)^2 + \left(dx^2\right)^2 + h_3(x^i,z)\left(\delta z\right)^2 + h_4(x^i,z)\left(\delta y^4\right)^2 . \tag{8.1}$$

The Einstein equations (3.7) with the Ricci h-tensor (3.2) and energy momentum d-tensor (3.16), or (3.19), transforms into

$$\frac{\partial^2 g_1}{\partial (x^1)^2} - \frac{1}{2g_1}\left(\frac{\partial g_1}{\partial x^1}\right)^2 - 2\kappa\Upsilon_3^3 g_1 = 0 . \tag{8.2}$$

By introducing the coordinates $\chi^i = x^i/\sqrt{|\kappa\Upsilon_3^3|}$ for the Class 3 (2) d-metrics and the variable $p = g_1'/g_1$, where by 'prime' in this section is considered the partial derivative ∂/χ^2, the equation (8.2) transforms into

$$p' + \frac{p^2}{2} + 2\epsilon = 0 , \tag{8.3}$$

where the vacuum case should be parametrized for $\epsilon = 0$ with $\chi^i = x^i$ and $\epsilon = 1(-1)$ for Class 2 (3) d-metrics. The equations (8.2) and (8.3) are, correspondingly, equivalent to the equations (7.6) and (7.8) with that difference that in this section we are dealing with 4D coefficients and values. The solutions for the h-metric are parametrized like (7.9), (7.10), and (7.11) and the coefficient $g_1(\chi^i)$ is given by a similar to (7.12) formula (for simplicity, here we elucidate the case $\epsilon < 0$) which for $p_{(0)}\left(\chi^1\right) = 0$ transforms into

$$g_1\left(\chi^1,\chi^2\right) = g_{1(0)}\left(\chi^1\right)\ \cos^2[\chi^2 - \chi^2_{(0)}\left(\chi^1\right)], \tag{8.4}$$

where $g_1\left(\chi^1\right), \chi^2_{(0)}\left(\chi^1\right)$ and $p_{(0)}\left(\chi^1\right)$ are some functions of necessary smoothness class on variable $\chi^1 = x^1/\sqrt{\kappa\varepsilon}$, ε is the energy density. The coefficients $g_1\left(\chi^1,\chi^2\right)$ (8.4) and $g_2\left(\chi^1,\chi^2\right) = 1$ define a h-metric of Class 3 (3.17) with energy-momentum d-tensor (3.19). The next step is the construction of h-components of d-metrics for different classes of symmetries of anisotropies.

Now, let us consider the system of equations (3.8) with the vertical Ricci d-tensor component (3.3) which are satisfied by arbitrary functions

$$h_3 = a_3(\chi^i) \text{ and } h_4 = a_4(\chi^i) . \tag{8.5}$$

If v-metrics depending on three coordinates are introduced, $h_a = h_a(\chi^i,z)$, the v-components of the Einstein equations transforms into (4.8) which reduces to (4.12) for prescribed values of $h_3(\chi^i,z)$, and, inversely, to (4.14) if $h_4(\chi^i,z)$ is prescribed. For h-metrics being conformally equivalent to (8.5) (see transforms (4.15)) we are dealing to equations of type (4.16) with integral varieties (4.17).

8.2 Rotation hypersurfaces horizons

We proof that there are static black hole and cylindrical like solutions of the Einstein equations with horizons being some 3D rotation hypersurfaces. The space components of corresponding d-metrics are conformally equivalent to some locally anisotropic deformations of the spherical symmetric Schwarzschild and cylindrical Weyl solutions. We note that for some classes of solutions the local anisotropy is contained in non-perturbative anholonomic structures.

8.2.1 Rotation ellipsoid configuration

There are two types of rotation ellipsoids, elongated and flattened ones. We examine both cases of such horizon configurations

Elongated rotation ellipsoid coordinates. An elongated rotation ellipsoid hypersurface is given by the formula [20]

$$\frac{\widetilde{x}^2+\widetilde{y}^2}{\sigma^2-1}+\frac{\widetilde{z}^2}{\sigma^2}=\widetilde{\rho}^2\,, \tag{8.6}$$

where $\sigma \geq 1$ and $\widetilde{\rho}$ is similar to the radial coordinate in the spherical symmetric case.

The space 3D coordinate system is defined

$$\begin{aligned} \widetilde{x} &= \widetilde{\rho}\sinh u \sin v \cos\varphi\,, \qquad \widetilde{y} = \widetilde{\rho}\sinh u \sin v \sin\varphi\,, \\ \widetilde{z} &= \widetilde{\rho}\ \cosh u \cos v\,, \end{aligned}$$

where $\sigma = \cosh u$, $(0 \leq u < \infty,\ 0 \leq v \leq \pi,\ 0 \leq \varphi < 2\pi)$. The hypersurface metric is

$$\begin{aligned} g_{uu} &= g_{vv} = \widetilde{\rho}^2\left(\sinh^2 u+\sin^2 v\right), \\ g_{\varphi\varphi} &= \widetilde{\rho}^2 \sinh^2 u \sin^2 v\,. \end{aligned} \tag{8.7}$$

Let us introduce a d-metric

$$\delta s^2 = g_1(u,v)du^2+dv^2+h_3\left(u,v,\varphi\right)\left(\delta t\right)^2+h_4\left(u,v,\varphi\right)\left(\delta\varphi\right)^2\,, \tag{8.8}$$

where δt and $\delta\varphi$ are N-elongated differentials.

As a particular solution (8.4) for the h-metric we choose the coefficient

$$g_1(u,v)=\cos^2 v\,. \tag{8.9}$$

The $h_3(u,v,\varphi) = h_3(u,v,\widetilde{\rho}\,(u,v,\varphi))$ is considered as

$$h_3(u,v,\widetilde{\rho}) = \frac{1}{\sinh^2 u+\sin^2 v}\frac{\left[1-r_g/4\widetilde{\rho}\right]^2}{\left[1+r_g/4\widetilde{\rho}\right]^6}\,. \tag{8.10}$$

In order to define the h_4 coefficient solving the Einstein equations, for simplicity, with a diagonal energy-momentum d-tensor for vanishing pressure, we must solve the equation (4.12) which transforms into a linear equation (4.13) if $\Upsilon_1 = 0$. In our case $s(u,v,\varphi) = \beta^{-1}(u,v,\varphi)$, where $\beta = (\partial h_4/\partial\varphi)/h_4$, must be a solution of

$$\frac{\partial s}{\partial \varphi} + \frac{\partial \ln\sqrt{|h_3|}}{\partial \varphi}\, s = \frac{1}{2}\,.$$

After two integrations (see [18]) the general solution for $h_4(u,v,\varphi)$, is

$$h_4(u,v,\varphi) = a_4(u,v)\exp\left[-\int_0^{\varphi} F(u,v,z)\,dz\right], \tag{8.11}$$

where

$$F(u,v,z) = (\sqrt{|h_3(u,v,z)|}[s_{1(0)}(u,v) + \frac{1}{2}\int_{z_0(u,v)}^{z}\sqrt{|h_3(u,v,z)|}dz])^{-1}, \tag{8.12}$$

$s_{1(0)}(u,v)$ and $z_0(u,v)$ are some functions of necessary smooth class. We note that if we put $h_4 = a_4(u,v)$ the equations (3.8) are satisfied for every $h_3 = h_3(u,v,\varphi)$.

Every d-metric (8.8) with coefficients of type (8.9), (8.10) and (8.11) solves the Einstein equations (3.7)–(3.10) with the diagonal momentum d-tensor

$$\Upsilon^{\alpha}_{\beta} = \mathrm{diag}\left[0,0,-\varepsilon = -m_0, 0\right],$$

when $r_g = 2\kappa m_0$; we set the light constant $c = 1$. If we choose

$$a_4(u,v) = \frac{\sinh^2 u\ \sin^2 v}{\sinh^2 u + \sin^2 v}$$

our solution is conformally equivalent (if not considering the time-time component) to the hypersurface metric (8.7). The condition of vanishing of the coefficient (8.10) parametrizes the rotation ellipsoid for the horizon

$$\frac{\widetilde{x}^2 + \widetilde{y}^2}{\sigma^2 - 1} + \frac{\widetilde{z}^2}{\sigma^2} = \left(\frac{r_g}{4}\right)^2,$$

where the radial coordinate is redefined via relation $\widetilde{r} = \widetilde{\rho}\left(1 + r_g/4\widetilde{\rho}\right)^2$. After multiplication on the conformal factor

$$\left(\sinh^2 u + \sin^2 v\right)\left[1 + \frac{r_g}{4\widetilde{\rho}}\right]^4,$$

approximating $g_1(u,v) = \sin^2 v \approx 0$, in the limit of locally isotropic spherical symmetry,

$$\widetilde{x}^2 + \widetilde{y}^2 + \widetilde{z}^2 = r_g^2,$$

the d-metric (8.8) reduces to

$$ds^2 = \left[1 + \frac{r_g}{4\widetilde{\rho}}\right]^4 \left(d\widetilde{x}^2 + d\widetilde{y}^2 + d\widetilde{z}^2\right) - \frac{\left[1 - r_g/4\widetilde{\rho}\right]^2}{\left[1 + r_g/4\widetilde{\rho}\right]^2} dt^2$$

which is just the Schwazschild solution with the redefined radial coordinate when the space component becomes conformally euclidean.

So, the d-metric (8.8), the coefficients of N-connection being solutions of (3.9) and (3.10), describe a static 4D solution of the Einstein equations when instead of a spherical symmetric horizon one considers a locally anisotropic deformation to the hypersurface of rotation elongated ellipsoid.

Flattened rotation ellipsoid coordinates. In a similar fashion we can construct a static 4D black hole solution with the horizon parametrized by a flattened rotation ellipsoid [20],

$$\frac{\widetilde{x}^2 + \widetilde{y}^2}{1 + \sigma^2} + \frac{\widetilde{z}^2}{\sigma^2} = \widetilde{\rho}^2 ,$$

where $\sigma \geq 0$ and $\sigma = \sinh u$.

The space 3D special coordinate system is defined

$$\widetilde{x} = \widetilde{\rho} \cosh u \sin v \cos \varphi , \qquad \widetilde{y} = \widetilde{\rho} \cosh u \sin v \sin \varphi ,$$
$$\widetilde{z} = \widetilde{\rho} \sinh u \cos v ,$$

where $0 \leq u < \infty,\ 0 \leq v \leq \pi,\ 0 \leq \varphi < 2\pi$.

The hypersurface metric is

$$g_{uu} = g_{vv} = \widetilde{\rho}^2 \left(\sinh^2 u + \cos^2 v\right) ,$$
$$g_{\varphi\varphi} = \widetilde{\rho}^2 \sinh^2 u \cos^2 v .$$

In the rest the black hole solution is described by the same formulas as in the previous subsection but with respect to new canonical coordinates for flattened rotation ellipsoid.

8.2.2 Cylindrical, bipolar and toroidal configurations

We consider a d-metric of type (8.1). As a coefficient for h-metric we choose

$$g_1(\chi^1, \chi^2) = \left(\cos \chi^2\right)^2$$

which solves the Einstein equations (3.7). The energy-momentum d-tensor is chosen to be diagonal, $\Upsilon^\alpha_\beta = \mathrm{diag}[0, 0, -\varepsilon, 0]$ with $\varepsilon \simeq m_0 = \int m_{(lin)} dl$, where $\varepsilon_{(lin)}$ is the linear 'mass' density. The coefficient $h_3(\chi^i, z)$ will be chosen in a form similar to (8.10),

$$h_3 \simeq \frac{\left[1 - r_g/4\widetilde{\rho}\right]^2}{\left[1 + r_g/4\widetilde{\rho}\right]^6}$$

for a cylindrical elliptic horizon. We parametrize the second v-component as $h_4 = a_4(\chi^1,\chi^2)$ when the equations (3.8) are satisfied for every $h_3 = h_3(\chi^1,\chi^2,z)$.

We note that in this work we only proof the existence of the mentioned type horizon configurations. The exact solutions and physics of so-called ellipsoidal black holes, black torus, black cylinders and black disks with, or not, local anisotropy will be examined in [34].

Cylindrical coordinates. Let us construct a solution of the Einstein equation with the horizon having the symmetry of ellipsoidal cylinder given by hypersurface formula [20]

$$\frac{\widetilde{x}^2}{\sigma^2} + \frac{\widetilde{y}^2}{\sigma^2 - 1} = \rho_*^2\,, \qquad \widetilde{z} = \widetilde{z}\,,$$

where $\sigma \geq 1$. The 3D radial coordinate $\widetilde{r}$ is to be computed from $\widetilde{\rho}^2 = \rho_*^2 + \widetilde{z}^2$.

The 3D space coordinate system is defined

$$\widetilde{x} = \rho_* \cosh u \cos v\,, \qquad \widetilde{y} = \rho_* \sinh u \sin v \sin\,, \qquad \widetilde{z} = \widetilde{z}\,,$$

where $\sigma = \cosh u$, $(0 \leq u < \infty,\ 0 \leq v \leq \pi)$.

The hypersurface metric is

$$g_{uu} = g_{vv} = \rho_*^2\left(\sinh^2 u + \sin^2 v\right), g_{zz} = 1\,. \qquad (8.13)$$

A solution of the Einstein equations with singularity on an ellipse is given by

$$h_3 = \frac{1}{\rho_*^2\left(\sinh^2 u + \sin^2 v\right)} \times \frac{\left[1 - r_g/4\widetilde{\rho}\right]^2}{\left[1 + r_g/4\widetilde{\rho}\right]^6}\,,$$
$$h_4 = a_4 = \frac{1}{\rho_*^2\left(\sinh^2 u + \sin^2 v\right)},$$

where $\widetilde{r} = \widetilde{\rho}\left(1 + r_g/4\widetilde{\rho}\right)^2$. The condition of vanishing of the time-time coefficient h_3 parametrizes the hypersurface equation of the horizon

$$\frac{\widetilde{x}^2}{\sigma^2} + \frac{\widetilde{y}^2}{\sigma^2 - 1} = \left(\frac{\rho_{*(g)}}{4}\right)^2, \qquad \widetilde{z} = \widetilde{z}\,,$$

where $\rho_{*(g)} = 2\kappa m_{(lin)}$.

By multiplying the d-metric on the conformal factor

$$\rho_*^2\left(\sinh^2 u + \sin^2 v\right)\left[1 + \frac{r_g}{4\widetilde{\rho}}\right]^4,$$

where $r_g = \int \rho_{*(g)} dl$ (the integration is taken along the ellipse), for $\rho_* \to 1$, in the local isotropic limit, $\sin v \approx 0$, the space component transforms into (8.13).

Bipolar coordinates. Let us construct 4D solutions of the Einstein equation with the horizon having the symmetry of the bipolar hypersurface given by the formula [20]

$$\left(\sqrt{\widetilde{x}^2+\widetilde{y}^2}-\frac{\widetilde{\rho}}{\tan\sigma}\right)^2+\widetilde{z}^2=\frac{\widetilde{\rho}^2}{\sin^2\sigma}\,,$$

which describes a hypersurface obtained under the rotation of the circles

$$\left(\widetilde{y}-\frac{\widetilde{\rho}}{\tan\sigma}\right)^2+\widetilde{z}^2=\frac{\widetilde{\rho}^2}{\sin^2\sigma}$$

around the axes Oz; because $|c\tan\sigma|<|\sin\sigma|^{-1}$, the circles intersect the axes Oz. The 3D space coordinate system is defined

$$\begin{aligned}
\widetilde{x} &= \frac{\widetilde{\rho}\sin\sigma\cos\varphi}{\cosh\tau-\cos\sigma}\,, \qquad \widetilde{y}=\frac{\widetilde{\rho}\sin\sigma\sin\varphi}{\cosh\tau-\cos\sigma}\,,\\
\widetilde{z} &= \frac{\widetilde{r}\sinh\tau}{\cosh\tau-\cos\sigma}\,,\\
&(-\infty<\tau<\infty, 0\le\sigma<\pi, 0\le\varphi<2\pi)\,.
\end{aligned}$$

The hypersurface metric is

$$g_{\tau\tau}=g_{\sigma\sigma}=\frac{\widetilde{\rho}^2}{(\cosh\tau-\cos\sigma)^2}, g_{\varphi\varphi}=\frac{\widetilde{\rho}^2\sin^2\sigma}{(\cosh\tau-\cos\sigma)^2}\,. \tag{8.14}$$

A solution of the Einstein equations with singularity on a circle is given by

$$h_3=\frac{[1-r_g/4\widetilde{\rho}]^2}{[1+r_g/4\widetilde{\rho}]^6} \qquad \text{and} \qquad h_4=a_4=\sin^2\sigma\,,$$

where $\widetilde{r}=\widetilde{\rho}(1+r_g/4\widetilde{\rho})^2$. The condition of vanishing of the time-time coefficient h_3 parametrizes the hypersurface equation of the horizon

$$\left(\sqrt{\widetilde{x}^2+\widetilde{y}^2}-\frac{r_g}{2}\,c\tan\sigma\right)^2+\widetilde{z}^2=\frac{r_g^2}{4\sin^2\sigma}\,,$$

where $r_g=\int\rho_{*(g)}dl$ (the integration is taken along the circle), $\rho_{*(g)}=2\kappa m_{(lin)}$.

By multiplying the d-metric on the conformal factor

$$\frac{1}{(\cosh\tau-\cos\sigma)^2}\left[1+\frac{r_g}{4\widetilde{\rho}}\right]^4\,, \tag{8.15}$$

for $\rho_*\to 1$, in the local isotropic limit, $\sin\upsilon\approx 0$, the space component transforms into (8.14).

Toroidal coordinates. Let us consider solutions of the Einstein equations with toroidal symmetry of horizons. The hypersurface formula of a torus is [20]

$$\left(\sqrt{\widetilde{x}^2+\widetilde{y}^2}-\widetilde{\rho}\; c\tanh\sigma\right)^2+\widetilde{z}^2=\frac{\widetilde{\rho}^2}{\sinh^2\sigma}\,.$$

The 3D space coordinate system is defined

$$\begin{aligned}
\widetilde{x} &= \frac{\widetilde{\rho}\sinh\tau\cos\varphi}{\cosh\tau-\cos\sigma}\,, \qquad \widetilde{y}=\frac{\widetilde{\rho}\sin\sigma\sin\varphi}{\cosh\tau-\cos\sigma}\,,\\
\widetilde{z} &= \frac{\widetilde{\rho}\sinh\sigma}{\cosh\tau-\cos\sigma}\,,\\
&(-\pi<\sigma<\pi, 0\leq\tau<\infty, 0\leq\varphi<2\pi)\,.
\end{aligned}$$

The hypersurface metric is

$$g_{\sigma\sigma}=g_{\tau\tau}=\frac{\widetilde{\rho}^2}{(\cosh\tau-\cos\sigma)^2}\,, \qquad g_{\varphi\varphi}=\frac{\widetilde{\rho}^2\sin^2\sigma}{(\cosh\tau-\cos\sigma)^2}. \tag{8.16}$$

This, another type of solution of the Einstein equations with singularity on a circle, is given by

$$h_3=\frac{[1-r_g/4\widetilde{\rho}]^2}{[1+r_g/4\widetilde{\rho}]^6} \qquad \text{and} \qquad h_4=a_4=\sinh^2\sigma\,,$$

where $\widetilde{r}=\widetilde{\rho}\,(1+r_g/4\widetilde{\rho})^2$. The condition of vanishing of the time-time coefficient h_3 parametrizes the hypersurface equation of the horizon

$$\left(\sqrt{\widetilde{x}^2+\widetilde{y}^2}-\frac{r_g}{2\tanh\sigma}c\right)^2+\widetilde{z}^2=\frac{r_g^2}{4\sinh^2\sigma}\,,$$

where $r_g=\int\rho_{*(g)}dl$ (the integration is taken along the circle), $\rho_{*(g)}=2\kappa m_{(lin)}$.

By multiplying the d-metric on the conformal factor (8.15), for $\rho_*\to 1$, in the local isotropic limit, $\sin v\approx 0$, the space component transforms into (8.16).

8.3 Disks with local anisotropy

The d-metric is of type (8.8)

$$\delta s^2=g_1(\rho,\zeta)d\rho^2+d\zeta^2+h_3\left(\rho,\zeta,\varphi\right)(\delta t)^2+h_4\left(\rho,\zeta,\varphi\right)(\delta\varphi')^2\,, \tag{8.17}$$

where the 4D coordinates are parametrized $x^1=\rho$ (the coordinate radius), $x^2=\zeta, y^3=t, y^4=\widetilde{\varphi}$ (like for the disk solution in general relativity [28]) and δt and $\delta\widetilde{\varphi}$ are N-elongated differentials). One uses also primed coordinates given with respect to corotating frame of reference, $\rho'=\rho, \zeta'=\zeta, \varphi'=\varphi-\Omega t, t'=t$, where Ω is the angular velocity as measured by an observer at ∞. The h-coordinates run respectively values $0\leq\rho<\infty$ and $-\infty<\zeta<\infty$. We consider a disk defined by conditions $\zeta=0$ and $\rho\leq\rho_0$.

As a particular solution (8.4) for the h-metric we choose the coefficient

$$g_1(\rho,\zeta) = (\cos\zeta)^2 \ . \tag{8.18}$$

The explicit form of coefficients h_3 and h_4 are defined by using functions

$$A = \rho^2 \exp[2\,(U-k)]\,, \quad B = -\exp(4U) \quad \text{and} \quad \widetilde{\varphi} = \varphi - \frac{Ba}{A-Ba^2}t\,, \tag{8.19}$$

where $U, k,$ and a are some functions on (ρ,ζ,φ). For the locally isotropic disk solutions we consider only dependencies on (ρ,ζ), in this case we shall write

$$U = U_0(\rho,\zeta)\,, \quad k = k_0(\rho,\zeta)\,, \quad a = a_0(\rho,\zeta) \text{ and } A = A_0(\rho,\zeta)\,, \quad B = B_0(\rho,\zeta)\,,$$

where the values with the index 0 are computed by using the Neugebauer and Meinel disk solution [28]. The (energy) mass density is taken

$$\varepsilon(\rho,\zeta,\varphi) = \delta(\zeta)\exp(U-k)\,\sigma_p(\rho,\varphi)\,,$$

where $\sigma_p(\rho,\varphi)$ is the (proper) surface mass density which is (in the anisotropic case) non-uniformly distributed on the disk; for locally isotropic distributions $\sigma_p = \sigma_p(\rho)$. After the problem is solved one computes σ_p as

$$\sigma_p = \frac{1}{2\pi}e^{U-k}\frac{\partial U'}{\partial\zeta}\;|_{\zeta=0^+} \ .$$

The time-time component h_3 is chosen in the form

$$h_3(\rho,\zeta,\varphi) = -\frac{AB}{A-Ba^2}$$

and, for simplicity, we state the second v-component of d-metric h_4 to depend only h-coordinates as

$$h_4 = a_4(\rho,\zeta) = A_0 - B_0\;a_0^2\,.$$

As in the locally isotropic case one introduces the complex Ernst potential

$$f(\rho,\zeta,\varphi) = e^{2U(\rho,\zeta,\varphi)} + ib(\rho,\zeta,\varphi)\,,$$

which depends additionally on coordinate φ. If the real and imaginary part of this potential are defined the coefficients (8.19) are computed

$$a(\rho,\zeta,\varphi) = \int_0^\rho \tilde{\rho}e^{-4U}b_{,\zeta}\,d\tilde{\rho}\,,$$

$$k(\rho,\zeta,\varphi) = \int_0^\rho \tilde{\rho}\Big[U_{,\tilde{\rho}}^2 - U_{,\zeta}^2 + \frac{1}{4}e^{-4U}(b_{,\tilde{\rho}}^2 - b_{,\zeta}^2)\Big]d\tilde{\rho}\,.$$

(In the integrands, one has $U = U(\tilde{\rho},\zeta,\varphi)$ and $b = b(\tilde{\rho},\zeta,\varphi)$.)

The Ernst potential is computed as in the locally isotropic limit with that difference that the values also depend on angular parameter φ,

$$f=\exp\left\{\int\limits_{K_1}^{K_a}\frac{K^2dK}{Z}+\int\limits_{K_2}^{K_b}\frac{K^2dK}{Z}-v_2\right\}, \tag{8.20}$$

with

$$Z=\sqrt{(K+iz)(K-i\bar{z})(K^2-K_1^2)(K^2-K_2^2)}\,,$$

$$K_1=\rho_0\sqrt{\frac{i-\mu}{\mu}}\quad(\mathrm{Re}\,K_1<0),\quad K_2=-\bar{K}_1\,,$$

where Re denotes the real part. The real (positive) parameter μ is given by

$$\mu=2\Omega^2\rho_0^2e^{-2V_0}\,,$$

where $V_0=$ const. The upper integration limits K_a and K_b in (8.20) are calculated from

$$\int\limits_{K_1}^{K_a}\frac{dK}{Z}+\int\limits_{K_2}^{K_b}\frac{dK}{Z}=v_0,\quad\int\limits_{K_1}^{K_a}\frac{KdK}{Z}+\int\limits_{K_2}^{K_b}\frac{KdK}{Z}=v_1\,, \tag{8.21}$$

where the functions v_0, v_1 and v_2 in (8.21) and (8.20) are given by

$$v_0=\int\limits_{-i\rho_0}^{+i\rho_0}\frac{H}{Z_1}dK,\quad v_1=\int\limits_{-i\rho_0}^{+i\rho_0}\frac{H}{Z_1}KdK\,, \tag{8.22}$$

$$v_2=\int\limits_{-i\rho_0}^{+i\rho_0}\frac{H}{Z_1}K^2dK\,,$$

$$H=\frac{\mu\ln\left[\sqrt{1+\mu^2(1+K^2/\rho_0^2)^2}+\mu(1+K^2/\rho_0^2)\right]}{\pi i\rho_0^2\sqrt{1+\mu^2(1+K^2/\rho_0^2)^2}}$$

$$(\mathrm{Re}\,H=0)\,,$$

$$Z_1=\sqrt{(K+iz)(K-i\bar{z})}\,, \tag{8.23}$$

where $\mathrm{Re}\,Z_1<0$ for ρ and ζ outside the disk. In (8.22) one has to integrate along the imaginary axis. The integrations from K_1 to K_a and K_2 to K_b in (8.20) and (8.21) have to be performed along the same paths in the two-seethed Riemann surface associated with $Z(K)$. The problem of finding K_a and K_b from (8.21) is a special case of Jacobi's inversion problem.

So, we have constructed a locally anisotropic generalization of the Neugebauer-Meinel [28] disk solution in general relativity, with an additional dependence on angle φ. In the locally isotropic limit, $g_1=(\cos\zeta)^2\approx1$, when the d-metric (8.17) is conformally equivalent (with the factor $\exp[2(U_0(\rho,\zeta)-k_0(\rho,\zeta))]$) to the disk solution from [28]).

8.4 Locally anisotropic generalizations of the Schwarzschild and Kerr solutions

8.4.1 A Schwarzschild like anisotropic solution

The d-metric of type (8.8) is taken

$$\delta s^2 = g_1(\chi^1,\theta)d(\chi^1)^2 + d\theta^2 + h_3\left(\chi^1,\theta,\varphi\right)(\delta t)^2 + h_4\left(\chi^1,\theta,\varphi\right)(\delta\varphi)^2\,, \qquad (8.24)$$

where on the horizontal subspace $\chi^1 = \rho/r_a$ is the undimensional radial coordinate (the constant r_a will be defined below), $\chi^2 = \theta$ and in the vertical subspace $y^3 = z = t$ and $y^4 = \varphi$. The energy-momentum d-tensor is taken to be diagonal $\Upsilon^\alpha_\beta = \mathrm{diag}[0,0,-\varepsilon,0]$. The coefficient g_1 is chosen to be a solution of type (8.4), $g_1(\chi^1,\theta) = \cos^2\theta$. For $h_4 = \sin^2\theta$ and $h_3(\rho) = -[1-r_a/4\rho]^2/[1+r_a/4\rho]^6$, where $r = \rho(1+\frac{r_g}{4\rho})^2$, $r^2 = x^2+y^2+z^2$, $r_a \doteq r_g$ is the Schwarzschild gravitational radius, the d-metric (8.24) describes an anisotropic solution of the Einstein equations which is conformally equivalent, with the factor $\rho^2\left(1+r_g/4\rho\right)^2$, to the Schwarzschild solution (written in coordinates (ρ,θ,φ,t)), embedded into a anisotropic background given by non-trivial values of $q_i(\rho,\theta,t)$ and $n_i(\rho,\theta,t)$. In the anisotropic case we can extend the solution for anisotropic (on angle θ) gravitational polarizations of point particles masses, $m = m(\theta)$, for instance in elliptic form, when

$$r_a(\theta) = \frac{r_g}{(1+e\cos\theta)}$$

induces an ellipsoidal dependence on θ of the radial coordinate,

$$\rho = \frac{r_g}{4\,(1+e\cos\theta)}\,.$$

We can also consider arbitrary solutions with $r_a = r_a(\theta,t)$ of oscillation type, $r_a \simeq \sin(\omega_1 t)$, or modelling the mass evaporation, $r_a \simeq \exp[-st], s = \mathrm{const} > 0$.

So, fixing a physical solution for $h_3(\rho,\theta,t)$, for instance,

$$h_3(\rho,\theta,t) = -\frac{\left[1 - r_a\exp[-st]/4\rho\,(1+e\cos\theta)\right]^2}{\left[1 + r_a\exp[-st]/4\rho\,(1+e\cos\theta)\right]^6}\,,$$

where $e = \mathrm{const} < 1$, and computing the values of $q_i(\rho,\theta,t)$ and $n_i(\rho,\theta,t)$ from (3.9) and (3.10), corresponding to given h_3 and h_4, we obtain an anisotropic generalization of the Schwarzschild metric.

We note that fixing this type of anisotropy, in the locally isotropic limit we obtain not just the Schwarzschild metric but a conformally transformed one, multiplied on the factor $1/\rho^2\left(1+r_g/4\rho\right)^4$.

8.4.2 A Kerr like anisoropic solution

The d-metric is of type (8.17) is taken

$$\delta s^2 = g_1(r/r_g,\theta)\,dr^2 + d\theta^2 + h_3(r/r_g,\theta,\widetilde{\varphi})\,(\delta t)^2 + h_4(r/r_g,\theta,\widetilde{\varphi})\,(\delta\widetilde{\varphi})^2\ . \qquad (8.25)$$

In the locally isotropic limit this metric is conformally equivalent to the Kerr solution, with the factor $r_\circ^2 = r^2 + a_\circ^2$, $a_\circ = \text{const}$ is associated to the rotation momentum, if

$$g_1^{[i]} = 1/\triangle(r)\,,$$

where $\triangle(r) = r^2 - rr_g + a_\circ^2$, r_g is the gravitational radius and the index $[i]$ points to locally isotropic values,

$$h_4^{[i]} = A(r,\theta)\,,$$

where

$$A(r,\theta) = \frac{\sin^2\theta}{r_\circ^2}\left(r^2 + a_\circ^2 + \frac{rr_g a_\circ^2}{r_\circ^2}\sin^2\theta\right) \text{ and } h_3^{[i]} = -\left(\frac{Q^2}{A} + B\right),$$

$$Q = \frac{rr_g a_\circ}{r_\circ^4}\sin^2\theta \text{ and } B = \frac{1}{r_\circ^2}\left(1 - \frac{rr_g}{r_\circ^2}\right).$$

The tilded angular variable $\widetilde{\varphi}$ is introduced with the aim to get a diagonal d-metric, $\widetilde{\varphi} = \varphi - \frac{Q}{A}t$.

A locally anisotropic generalization is to be found if we consider, for instance, that $r_g \to r_g(\theta)$ is defined by an anisotropic mass $\tilde{m}(\theta)$ and the locally isotropic values with $r_g = cons$ are changed into those with variable $r_g(\theta)$. The d-metric coefficient $h_4(r,\theta,\widetilde{\varphi})$ and the corresponding N-connection components are taken as to solve the equations (3.8) and (3.9).

9. Some additional examples

9.1 Two-soliton locally anisotropic solutions

Instead of one soliton solutions we can also consider anisotropic deformations of multi-soliton configurations (as a review see [30]). In this Subsection we give an example of anholonomic, for simplicity, two-soliton configuration in general relativity. The d-metric to be constructed is of Class 1 with the h-component being a solution (type (4.4)) of the equation (4.3) for $\epsilon = -1$ and $x^i = (t,x)$.

The horizontal component of d-metric is induced, via a conformal transform (5.1), from the so-called two soliton 2D Lorentz metric (here we follow the denotations from [10] being adapted to locally anisotropic constructions)

$$d\widetilde{s}^2 = -2\frac{FG}{F^2+G^2}dt^2 + \frac{G^2-F^2}{F^2+G^2}dx\,,$$

where for the two soliton solution

$$F = \cot\mu \sinh\left[\widetilde{m}\sin\mu\gamma\left(t+vx\right)\right],$$
$$G = \sinh\left[\widetilde{m}\cos\mu\gamma\left(x-vt\right)\right]$$

or, for the soliton-anti-soliton solution,

$$F = \cot\mu \cosh\left[\widetilde{m}\sin\mu\gamma\left(t+vx\right)\right],$$
$$G = \cosh\left[\widetilde{m}\cos\mu\gamma\left(x-vt\right)\right],$$

with constant parameters μ, γ and v. The function

$$u\left(t,x\right) = 4\tan^{-1}\left(F/G\right)$$

is a solution of 2D euclidean sine-Gordon equation

$$\partial_t^2 u + \partial_x^2 u = \widetilde{m}^2 \sin u\,.$$

The locally anisotropic deformation is described by an anisotropic dilaton field $\omega\left(x^i\right)$ chosen to solve the Poisson equation (5.3) with the source ρ (4.2) computed by using the two soliton function $u\left(t,x\right)$. In consequence, the h-metric is of the form

$$g = \exp[\omega\left(x^i\right)] \times \left[-2\frac{FG}{F^2+G^2}dt^2 + \frac{G^2-F^2}{F^2+G^2}\right]dx\,. \tag{9.1}$$

The next step is the construction of a soliton like v-metric. Let, for simplicity,

$$h_3 = a_3\left(x^i\right) = \exp[\omega\left(x^i\right)]\frac{G^2-F^2}{F^2+G^2} \tag{9.2}$$

and $h_4 = h_4\left(x^i, z\right)$ is to be defined by the equation (4.8), which for $\partial h_3/\partial z = 0$ transforms into

$$h_4\frac{\partial^2 h_4}{\partial z^2} - \frac{1}{2}\left(\frac{\partial h_4}{\partial z}\right)^2 - 2k\Upsilon_1 a_3\left(x^i\right)\left(h_4\right)^2 = 0\,, \tag{9.3}$$

where we consider a diagonal energy-momentum d-tensor $\Upsilon_\beta^\alpha = \mathrm{diag}[-\varepsilon, 0, 0, 0]$. Introducing a new variable $h_4 = \xi^2$, the equation (9.3) transform into a linear second order differential equation on z when coordinates x^i are treated as parameters,

$$\frac{\partial^2 \xi}{\partial z^2} + \lambda\left(x^i\right)\xi = 0\,,$$

where $\lambda\left(x^i\right) = 2\kappa\varepsilon a_3\left(x^i\right)$. The general solution $\xi(x^i, z)$ is

$$\xi = \begin{cases} c_1\cosh(z\sqrt{|\lambda|}) + c_2\sinh(z\sqrt{|\lambda|})\,, & \lambda < 0\,; \\ c_1 + c_2 z\,, & \lambda = 0\,; \\ c_1\cos(z\sqrt{\lambda}) + c_2\sin(z\sqrt{\lambda})\,, & \lambda > 0\,, \end{cases}$$

where c_1 and c_2 are some functions on x^i.

The coefficients $h_3 = a_3(x^i)$ (9.2) and $h_4 = \xi^2(x^i, z)$ define a h-metric induced by a 2D two soliton equation. The complete d-metric solving the Einstein equations (2.8) is defined by considering v-coefficients of type (9.1).

9.2 Kadomtsev-Petviashvily structures and non-diagonal energy-momentum d-tensors

Such structures, for diagonal energy-momentum d-tensors and vacuum Einstein equations, where proven to exist in subsection 4.2.2. Here we show that another type of three dimensional soliton structures could be generated by nondiagonal components Υ_{31} and Υ_{32}.

For $\Upsilon_1^1 = \Upsilon_2^2 = 0$ every function $h_4 = a_4(x^i)$ solves the v-component of Einstein equations (3.8). Let us consider a function $h_3 = h_3(x^i, z)$. If the anholonomic constraints on the system of reference are imposed by N-connection coefficients $N_i^3 = q_i$, when

$$q_i = -2\kappa\Upsilon_{3i}h_3\left[h_3\left(h_3^*\right)^2 + \epsilon\left(\dot{h}_3 + 6h_3h_3' + h_3'''\right)'\right]^{-1},$$

where $\epsilon = \pm 1$, the system of equations (3.9) reduces to the Kadomtsev-Petviashvili equation for h_3,

$$h_3^{**} + \epsilon\left(\dot{h}_3 + 6h_3h_3' + h_3'''\right)' = 0\,.$$

The solution of Einstein equations is to completed by considering some functions $N_i^4 = n_i$ satisfying (3.10) and a h-metric $g_{ij}(x^k)$ solving (3.7).

9.3 Anholonomic soliton like vacuum configurations

The main result of Belinski-Zakharov-Maison works [4, 24] was the proof that vacuum gravitational soliton like structures could be defined in the framework of general relativity with $h_{ab}(x^i)$ (from 4D metric (1.4)) being a solution of a generalized type of sine-Gordon equations. The function $f(x^i)$ (from (1.4)) is to be determined by some integral relations after the components $h_{ab}(x^i)$ have been constructed.

By reformulating the problem of definition of soliton like integral varieties of vacuum Einstein equations from the viewpoint of anholonomic frame structures, there are possible further generalizations and constructions of new classes of solutions.

For vanishing energy-momentum d-tensors the Einstein equations (3.7)–(3.10) transform into

$$2(g_1'' + \ddot{g}_2) - \frac{1}{g_2}\left(\dot{g}_2^2 + g_1'g_2'\right) - \frac{1}{g_1}\left({g_1'}^2 + \dot{g}_1\dot{g}_2\right) = 0\,; \tag{9.4}$$

$$h_4^{**} - \frac{1}{2h_4}(h_4^*)^2 - \frac{1}{2h_3}h_3^*h_4^* = 0\,; \tag{9.5}$$

$$q_1\left[\frac{h_3^{**}}{h_3} - \frac{(h_4^*)^2}{2h_4} - \frac{h_3^*h_4^*}{2h_3}\right] + \frac{h_4^*}{2}\left(\frac{\dot{h}_3}{h_3} + \frac{\dot{h}_4}{h_4}\right) - \dot{h}_4^* = 0\,, \tag{9.6}$$

$$
\begin{aligned}
q_2\left[\frac{h_3^{**}}{h_3}-\frac{h_4^{*}}{2h_4}-\frac{h_3^{*}h_4^{*}}{2h_3}\right]+\frac{h_4^{*}}{2}\left(\frac{h_3'}{h_3}+\frac{h_4'}{h_4}\,h_4^{*}\right)-{h_4'}^{*}&=0\,;\\
-n_1^{**}+\left(\frac{h_4}{h_3}h_3^{*}-\frac{3}{2}h_4^{*}\right)n_1^{*}&=0\,,\\
-n_2^{**}+\left(\frac{h_4}{h_3}h_3^{*}-\frac{3}{2}h_4^{*}\right)n_2^{*}&=0\,,
\end{aligned} \tag{9.7}
$$

where we suppose that g_1, g_2, h_3 and h_4 are not zero.

The equation (9.4), relates two components and their first and second order partial derivatives of a diagonal h-metric $g_1(x^i)$ and $g_2(x^i)$. We can prescribe one of the components in order to find the second one by solving a second order partial differential equation. For instance, we can consider the h-metric to be induced by a soliton-dilaton solution (like in the section 4, but for vacuum solitons the constants will be not defined by any components of the energy-momentum d-tensor).

Let us fix a soliton 2D solution with diagonal auxiliary metric

$$
\begin{aligned}
\widetilde{g}_{ij} &= \mathrm{diag}\{\widetilde{g}_1=\epsilon\sin^2\left[v\left(x^i\right)/2\right],\widetilde{g}_2=\cos^2\left[v\left(x^i\right)/2\right]\},\\
\epsilon &= \pm 1\,,
\end{aligned}
$$

and model the local anisotropy by an anisotropic dilaton field $\omega\left(x^i\right)$ relating the metric $\widetilde{g}_{ij}$ with the h-components g_{ij} via a conformal transform (5.1). The anisotropic dilaton is to be found as a solution of the equations (5.3) where the source $\rho\left(x^i\right)$ is computed by using the formula (4.2).So, we conclude that vacuum h-metrics can be described by corresponding soliton-dilaton systems.

The equation (9.5) relates two components and their first and second order partial derivatives on z of a diagonal v-metric $h_3(x^i,z)$ and $h_4(x^i,z)$ which depends on three variables. We also can prescribe one of these components (for instance, as was shown in details in section 4.2 to be a solution of the Kadomtsev-Patviashvili, or (2+1) dimensional sine-Gordon equation; the Belinski-Zakharov-Maison solutions can be considered as some particular case soliton vacuum configurations which do not depend on variable z) the second v-component being defined after solution of the resulted partial differential equation on z, with the h-coordinates x^i treated as parameters.

If the values $h_3(x^i,z)$ and $h_4(x^i,z)$ are defined, we have algebraic equations (9.6) for calculation of coefficients $q_1(x^i,z)$ and $q_2(x^i,z)$. The equations (9.7) are satisfied by arbitrary $n_1(x^i,z)$ and $n_2(x^i,z)$ depending linearly on the third variable z.

10. Conclusions

In this paper, we have elaborated a new method of construction of exact solutions of the Einstein equations by using anholonomic frames with associated nonlinear connection structures.

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We analyzed 4D metrics

$$ds^2 = g_{\alpha\beta}\ du^\alpha du^\beta$$

when $g_{\alpha\beta}$ are parametrized by matrices of type

$$\begin{bmatrix} g_1 + {q_1}^2 h_3 + {n_1}^2 h_4 & q_1 q_2 h_3 + n_1 n_2 h_4 & q_1 h_3 & n_1 h_4 \\ +q_1 q_2 h_3 + n_1 n_2 h_4 & g_2 + {q_2}^2 h_3 + {n_2}^2 h_4 & q_2 h_3 & n_2 h_4 \\ q_1 h_3 & q_2 h_3 & h_3 & 0 \\ n_1 h_4 & n_2 h_4 & 0 & h_4 \end{bmatrix} \tag{10.1}$$

with coefficients being some functions of necessary smooth class $g_i = g_i(x^j), q_i = q_i(x^j, z), n_i = n_i(x^j, z)$, $h_a = h_a(x^j, z)$. Latin indices run respectively $i, j, k, \ldots = 1, 2$ and $a, b, c, \ldots = 3, 4$ and the local coordinates are denoted $u^\alpha = (x^i, y^3 = z, y^4)$. A metric (10.1) can be diagonalized,

$$\delta s^2 = g_i(x^j) \left(dx^i\right)^2 + h_a(x^j, z) \left(\delta y^a\right)^2 \ ,$$

with respect to anholonomic frames (1.9) and (1.10), here we write down only the 'elongated' differentials

$$\delta z = dz + q_i(x^j, z) dx^i\ ,\ \ \delta y^4 = dy^4 + n_i(x^j, z) dx^i\ .$$

The key result of this paper is the proof that for the introduced ansatz the Einstein equations simplify substantially for 3D and 4D spacetimes, the variables being separated:

- The equation (3.7) with the non-trivial component of the Ricci tensor (3.2) relates two (so-called, horizontal) components of metric g_i with the (so-called, vertical) values of the diagonal energy-momentum tensor. We proved that such components of metric could be described by soliton-dilaton and black hole like solutions with parameters being determined by vertical sources.

- Similarly, the equation (3.8) with the non-trivial component of the Ricci tensor (3.2) relates two vertical components of metric h_a with the horizontal values of the diagonal energy-momentum tensor. The vertical coefficients of metric could depend on three variables (x^i, z) and this equation contains their first and second derivatives on z, the dependence on horizontal coordinates x^i being parametric.

- As to the rest of equations (3.9) and (3.10) with corresponding non-trivial Ricci tensors (3.4) and (3.6), they form an algebraic system for definition of the nonlinear connection coefficients $q_i(x^i, z)$ and second order differential equation on z for the nonlinear connection coefficients $n_i(x^i, z)$ after the functions $h_a(x^i, z)$ have been defined and non-diagonal components of energy-momentum tensor are given.

The Einstein equations consist a system of second order nonlinear partial differential equations whose particular solutions are selected from the general integral variety by imposing some physical motivated conditions on the type of singularities, horizon hypersurfaces, perturbative and/or non-perturbative behavior of background configurations, limit correspondences with some well known solutions, physical laws, symmetries and so on.

We investigated the conditions when from the class of solutions of 4D and 3D gravitational field equations parametrized by metric ansatzs of type (10.1) we can obtain some locally anisotropic generalizations of well known soliton-dilaton, black hole, cylinder and disk solutions.

In this paper we have shown that one can use solutions of generalized sine-Gordon equations in two and three dimensions to generate 4D solutions of Einstein gravity with soliton-dilaton parameters being related to 4D energy-momentum values. We have found a broad class of 2D, 3D and 4D black hole configurations with generic local anisotropy. Our results seem to indicate that there is a deep connection between black hole and soliton-dilaton states in gravitational theories of lower and 4D dimensions. Via nonlinear superpositions the lower dimensional locally anisotropic configurations induce similar structures in higher dimensions. We conclude that if the former direct applications of the 2D soliton-dilaton-black hole models (more naturally treated in the framework of 2D gravity and string theory) are very rough approximations for general relativity, after introducing of some well defined principles of nonlinear superposition, the lower dimensional solutions could be considered as some building blocks for construction of non-perturbative solutions in four dimensions.

We presented a series of computations involving the dynamics of locally anisotropic gravitational soliton deformations, black hole dynamics and constructed exact 4D and 3D solutions of the Einstein equations with horizons being (under corresponding dimension) of elliptic, rotation ellipsoidal, bipolar, elliptic cylinder and toroidal configuration. We showed that such solutions are naturally contained in general relativity and defined by corresponding anholonomic constraints, anisotropic distributions of masses and energy densities and could model some anisotropic nonlinear self-gravitational polarizations and renormalizations of gravitational and cosmological constants. It should be emphasized that for our static black hole locally anisotropic solutions does not holds restrictions of type stated by Israel's [9] uniqueness theorems of the Schwarzshild metric among all static vacuum black holes configurations (see a detailed discussion of the problem in [14]). Our static locally anisotropic solutions, having deformed horizons, transform just into some metrics being conformally equivalent to the Schwarzshild solution (for ellipsoidal symmetries) at large distances, or have a different topology (for torus like symmetries; our class of solutions for toroidal black holes, constructed by using anholonomic frames, differs from those defined, for instance, in refs. [21, 22] by some exotic configurations of matter). In the anisotropic cases there are also additional dependencies on anisotropy coordinates.

Our approach represents just a first step in the differential geometric and nonlinear analysis of the role that solitons and singular configurations with local anisotropy plays in 2D, 3D and 4D gravity. The natural developments of our approach would be to use nonlinear superpositions to describe the semiclassical and quantum dynamics of extremal black holes induced from string theory, the corresponding nonequilibrium thermodynamics of such black holes. One would be of interest supersymmetric extensions of the method and investigation of the mentioned non-perturbative structures in the framework of string theory. Work is in progress to address these issues.

Acknowledgments

The author thanks Prof. Dr. H. Dehnen for hospitality and support during his DAAD Fellowship at Konstanz University, Germany. He is also grateful for constructive discussions to Prof. G. Neugebauer and Dr. R. Meinel.

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